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Elastic stability of the top chord of a three-span continuous pony truss bridge

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ELASTIC STABILITY OF THE TOP CHORD OF A THREE-SPAN CONTINUOUS PONY TRUSS BRIDGE

by

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INTRODUCTION

Object

The object of this investigation was to make an analytical study of the elastic stability of the top chord of a three-span continuous pony truss bridge. The particular bridge studied was designed by the Iowa Highway Commission in 1947 and erected over the North River on State Highway 60 a few miles southeast of Des Moines, Iowa.

The present Standard Highway Specifications of the American Association of State Highway Officials (1) require that a check be made for stability of the top chord of a half-through or pony truss bridge. These Specifications suggest Timoshenko (2) as a reference for a procedure of analysis for stability; in this reference, Timoshenko, however, does not cover the case of a continuous structure in which the top chord will have members in tension as well as in compression.

In a previous edition of the Standard Highway Specifications (3, p. 173) it was stated that:

> The vertical truss members and the floorbeam connections of half-through truss spans shall be proportioned to resist a lateral force, applied at the top chord panel points of the truss, determined by the following equation:

$R = 150(A + P)$ in which:
R = lateral force in pounds.
A = area of cross section of the chord in square inches.
P = panel length in feet.

This is an empirical procedure and no additional check for stability was specified as being required. This same procedure was employed by the Iowa Highway Commission for the design of the pony truss used as an example to which the results of this thesis were applied.

The problem studied was one of a long top chord which is essentially a beam-column elastically supported at intermediate points by a frame composed of the verticals of the truss and the floor beams framing into the verticals. The type of solution was limited to one in which it was assumed that the truss members were in a plane before and after the loads were applied to the structure. The effect of the diagonal web members of the truss on the stability of the top chord was neglected. Should the influence of the diagonal web members be desired, it could be introduced by the method developed in this thesis after careful consideration is taken of the effect of the gusset plates. The problem then was limited to the stability analysis of a member composed of the top chord and end posts of a three-span continuous pony truss bridge; the top chord is elastically supported at the panel points against deflections in a direction at right angles to the plane of the truss, and it is assumed that the members form a straight line when viewed from above.

Briefly, the procedures of analysis used in this investigation to check stability are: (1) When an external clockwise moment, under a condition of equilibrium, is required at a joint to rotate the joint in a clockwise direction, there is an indication of stability; but, when the external moment necessary to hold the joint in equilibrium is opposite to the direction of the rotation, then there is an indication of instability. And, (2) When a force, under a condition of equilibrium, acting out of the plane of the truss is applied at a joint and when the direction of the deflection of the joint is in the direction of the applied force, there is an indication of stability; but, when, under a condition of equilibrium the direction of the force must be opposite to the direction of the deflection of the joint, then a condition of instability is indicated.

In order to solve a problem involving stability it is necessary to deal with a failure load; this leads to the use of the term "load factor" instead of "factor of safety". The term "load factor" has been used in the aeronautical field and is coming into use in the civil engineering structural field. The "load factor" as it applies to a buckling problem can be defined as the factor which when multiplied by the actual or design live load is expected to produce a load causing a buckling failure in the structure.

The actual procedure of determining the load factor for a particular structure is one of selecting trial values of the load factor; that is, a load factor is assumed and the structure checked for stability. This process is then repeated until the load factor which causes failure by buckling is found. The work in this thesis was limited to axial unit stresses that were within the yield strength of the material.

Historical Background

The procedures used in the analysis in this thesis were based on the series and stiffness criteria which are extensions of the conventional moment distribution type of solution as applied to continuous frames. The literature used as a background for the work is briefly reviewed.

The series criterion is an extension of the moment distribution procedure of Hardy Cross (4). In 1935, James (5) extended the Cross method of moment distribution to the analysis of continuous members subject to axial loads. Tables of stiffness and carry-over factors for structural members under axial load have been published by Lundquist and Kroll (6) (7). In 1937, Lundquist (8) applied the principles of moment distribution to stability problems and developed the series and stiffness criteria. In 1941, Hoff (9) gave the first rigorous proof of these criteria developed by Lundquist. The procedures for stability analysis described above were limited to cases of continuous members over rigid supports or trusses in which the buckling was considered in the plane of the truss; in other words, the joints of the trusses were considered to rotate but were assumed as fixed in position. In 1946, Hu and Libove (10) applied the principles of moment distribution to the stress analysis of an elastically supported beam-column. Winter, in 1948, (11) published a bulletin on buckling of trusses and rigid frames. In the analysis of the rigid frames he considered the effect of joint displacement, or sidesway, on the buckling stability of the frame. Also in 1948, Kavanagh (12) published a survey of the theory of framework stability that was made available to members of the Column Research Council. In 1949, Wessman and Kavanagh (13, p. 968) made a statement as follows:

> It will be of some interest however to know that additional investigation has demonstrated clearly that the buckling loads for steel buildings and bridge trusses, with members having slenderness ratios in accord with current design practice, are so close to the loads corresponding to the yield stress of the steel employed, that there is absolutely no need for buckling load analysis. The yield strength is the ceiling regardless of whether ordinary structural steel, low alloy steel, or silicon steel is used. This conclusion is not true, however, for structural aluminum alloys.

Whereas buckling of truss members in the plane of the truss was considered by Wessman and Kavanagh, in this thesis, buckling of the entire top chord out of the plane of the truss was considered. Budiansky, Seide, and Weinberger (14) developed a set of curves for the buckling of a column on equally spaced deflectional and rotational springs. Their results consider only the special case of equal spans, constant compressive force, equal deflectional springs, intermediate rotational springs of equal stiffness and end rotational springs of half the stiffness of the intermediate springs. Work on the buckling of the top chord of a single-span pony truss bridge has been carried on by Holt (15) (16).

NOTATION

Only those symbols which are used repeatedly throughout the thesis are included below with the number of the page on which the symbol is first used.

A_2	Substitution for $S_2 + S_{Bzz}$	(58)
B_2	Substitution for $\frac{S_2'''}{L_2} - S_{Byz}$	(58)
C	Carry-over factor	(13)
C_1	Carry-over factor	(32)
C_{a1R}	Carry-over factor	(36)
C_{b2R}	Carry-over factor	(41)
C_{c2R}	Carry-over factor	(46)
D_2	Substitution for $T_2 + S_{Byy}$	(58)
D.F.	Distribution factor	(48)
E	Modulus of elasticity	(13)
I	Moment of inertia	(13)
k	Substitution for $\sqrt{\frac{P}{EI}}$	(13)
L	Length of a member	(13)
L_1	Length of a member	(32)
M_A	Moment at the end of a member with no adjacent member	(13)
M_{AB}	Moment at end A of member AB	(32)

M_t	Torsional moment about the t-axis (77)
P	Axial stress (13)
P_1	Axial stress (33)
r	Stability factor (11)
R_{1R}	Rotation of a member (32)
S	Stiffness factor (13)
S_1	Stiffness factor (32)
S_{1R}	Stiffness factor (35)
S'''	Substitution for $S(1 + C)$ (19)
S_1'''	Substitution for $S_1(1 + C_1)$ (33)
S_{Bzz}	Rotational stiffness (51)
S_{Byy}	Translational stiffness (51)
S_{Byz}	Quantity used in tho stiffness criterion (51)
S'_{Byz}	Quantity used in the stiffness criterion (51)
S_t	Torsional stiffness (77)
t_B	Stiffness of an elastic support (32)
t'_{1L}	Translational stiffness (31)
T	Translational stiffness (16)
T_1	Translational stiffness (33)
T'_{1R}	Translational stiffness (34)
V_A	Shear at the end of a member with no adjacent member (13)
V_{AB}	Shear at end A of member AB (31)
x, y	Rectangular co-ordinates (13)
y_A	Deflection of a joint (24)

α	Rotation of a joint	(31)
θ_A	Rotation of a joint	(26)
δ	Deflection of a joint	(18)
ϕ_f	Substitution for $\frac{1}{(kL)^2}$ (kL Cos kL - 1)	(15)
ϕ_n	Substitution for $\frac{1}{(kL)^2}$ (1 - kL Cot kL)	(15)
ϕ_f'	Substitution for $-\frac{1}{(kL)^2}$ (kL Cosh kL - 1)	(21)
ϕ_n'	Substitution for $-\frac{1}{(kL)^2}$ (1 - kL Coth kL)	(21)

STABILITY ANALYSIS OF THE ELASTICALLY SUPPORTED BEAM-COLUMN

In this thesis the analysis was, of course, based on the works which have been cited. For purposes of verification and clarification, all equations used were derived anew even though various derivations for some of them can be found in the aforementioned literature. In many cases the derivations are of a more general nature and in some the derivations are entirely new. This complete set of derivations is essential for the clear and conclusive presentation of the final solution. The equations for the stiffness criterion were derived with a sign convention which is believed clearer than previously used signs. In addition, the stability criteria were applied to an existing three-span continuous pony truss bridge where before the application had been limited to single-span structures.

Series Criterion

The series criterion is one of the criteria used to check the stability in a structure. The series criterion states that when a converging series is obtained a condition of stability is indicated, but when a diverging series is obtained then a condition of instability is indicated. The series in this case is obtained from a moment distribution solution. The general procedure as applied to a beam-column elastically supported at the joints (representing the top chord of the pony truss) is as follows:

Apply an external moment at any joint, N, of the beam-column.

Balance joint N which is permitted to rotate and deflect while other joints are held against rotation but may deflect subject to the elastic restraint at each joint.

Then, joint N is fixed against rotation but is free to deflect subject to the elastic restraints contributed by all members and supports.

The foregoing operations will cause moments to be carried over to the ends of every member of the beam-column.

Now the other joints, except N, are balanced one at a time and moments carried over to the other members as already described.

This process is repeated until all of the joints are balanced except, of course, joint N. The unbalance at joint N is determined by adding all of the moments carried to joint N while balancing the other joints in the structure.

The unbalanced moment at N divided by the originally applied external moment gives a stability factor, r.

Now if joint N were balanced a second time and the entire process repeated, the new total moment carried over to N divided by the originally applied external moment would equal r^2. If this process were repeated a series of the type

$$1 + r + r^2 + r^3 + r^4 + \text{-----}$$

would be obtained. For this series to converge, r must be less than one. When r equals one, a condition of neutral equilibrium exists. The physical significance of neutral equilibrium can be explained as the condition when no external moment is required to rotate a joint of the structure. When r is less than one, a clockwise external moment is required to rotate the joint in a clockwise direction which has been defined as an indication of stability and when r is greater than one, a counterclockwise external moment, under a condition of equilibrium, is required for a clockwise rotation of the joint and this has been defined as an indication of instability.

Derivation of Formulas for the Series Criterion

For a moment distribution solution the elastic constants of the members are required. Two of these elastic constants are the stiffness and carry-over factors. When axial loads are neglected, these factors are a function of the physical properties of the member. When the effect of an axial load in the member is considered in the moment distribution method of analysis, the stiffness and carry-over factors are dependent on the magnitude and type, tension or compression, of axial force, as well as the physical properties of the members.

If a member is hinged against deflection at the near end and fixed against rotation and deflection at the far end, the ratio of the moment at the far end to the applied moment at the near end is known as the carry-over factor and will be denoted by C. Also, the moment necessary to produce a rotation of one radian of the near end is known as the stiffness factor and will be denoted by S. In the references taken from the aeronautical field the stiffness factor is defined as the moment that will produce a rotation of one-quarter radian. However, the definition of one radian rotation as usually used in the structural field will be used in this thesis.

The differential equation for the deflection of the member in Figure 1 is

$$EI \frac{d^2y}{dx^2} = -M_A + V_A x - Py \qquad (1)$$

in which $V_A = \dfrac{M_A + M_B}{L}$. Since P, E, and I are constants, let $k^2 = \dfrac{P}{EI}$. The solution of equation (1) is

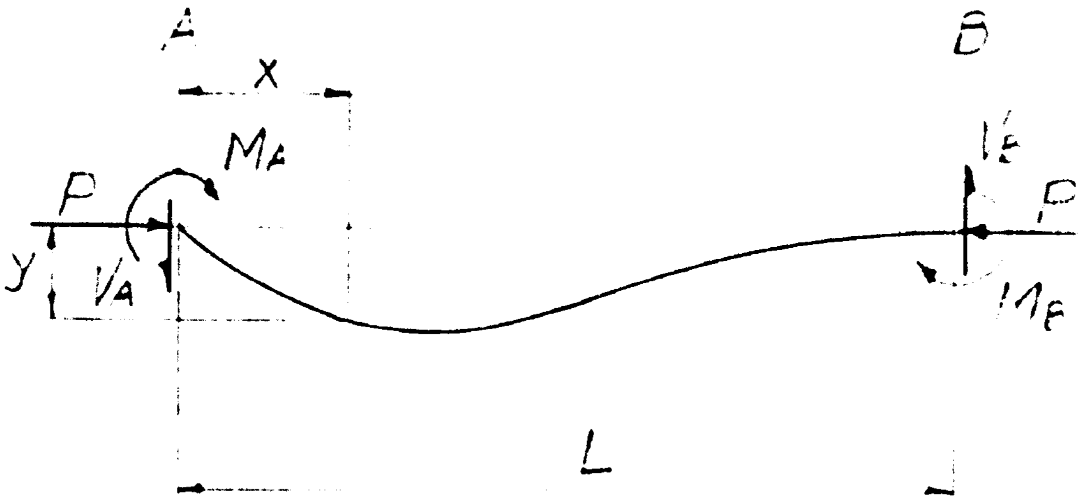


Figure 1. Compression member with its left end rotated.

$$y = C_1 \text{ Cos } kx + C_2 \text{ Sin } kx - \frac{M_A}{P} + \frac{M_A + M_B}{PL} x \qquad (2)$$

The constants C_1 and C_2 in equation (2) are determined from the boundary conditions, $x = 0$, $y = 0$, and $x = L$, $y = 0$ resulting in

$$C_1 = \frac{M_A}{P} \text{ and } C_2 = -\frac{1}{P \text{ Sin } kL} \left[M_A \text{ Cos } kL + M_B \right]$$

Therefore,

$$y = \frac{M_A}{P} \text{ Cos } kx - \frac{M_A \text{ Cos } kL + M_B}{P \text{ Sin } kL} \text{ Sin } kx$$

$$- \frac{M_A}{P} + \frac{M_A + M_B}{PL} x \qquad (3)$$

Taking the derivative of equation (3) with respect to x, introducing the boundary condition $x = L$, $\frac{dy}{dx} = 0$, and then making the substitutions

$$\phi_f = \frac{1}{(kL)^2} (kL \text{ Csc } kL - 1)$$

$$\phi_n = \frac{1}{(kL)^2} (1 - kL \text{ Cot } kL)$$

the carry-over factor is found to be

$$C = \frac{M_B}{M_A} = \frac{\phi_f}{\phi_n} \qquad (4)$$

The stiffness factor, which has been defined as M_A with the condition $x = 0$, $\frac{dy}{dx} = 1$, is found to be

$$S = M_A = \frac{EI}{L} \frac{\phi_n}{\phi_n^2 - \phi_f^2} \qquad (5)$$

Values of C have been published (6) (7) for values of kL. Values of $\frac{S}{\frac{EI}{L}}$ for values of kL have been tabulated in the same source in which the stiffness has been defined as the moment that causes a rotation of one-quarter radian. Therefore, the tabulated values can be used for the above case if they are equated to $\frac{S}{\frac{4EI}{L}}$. Approximate values of C and S can be obtained from the curves included in Appendix A. When the axial force is zero, kL equals zero and $\frac{S}{\frac{4EI}{L}}$ is equal to one. Therefore, for this special case $S = \frac{4EI}{L}$ which, of course, is recognized as the usual stiffness factor when axial forces are not present or are neglected. The translational stiffness of a member is another elastic constant required for the moment distribution solution for a beam-column type of member. The translational stiffness is defined as the shear necessary to deflect one end of the member a unit distance with respect to the other end when neither end is permitted to rotate. The translational stiffness of the member will be denoted by T.

The differential equation for the deflection of the member in Figure 2 is

$$EI \frac{d^2y}{dx^2} = - Py + M_A - V_A x \qquad (6)$$

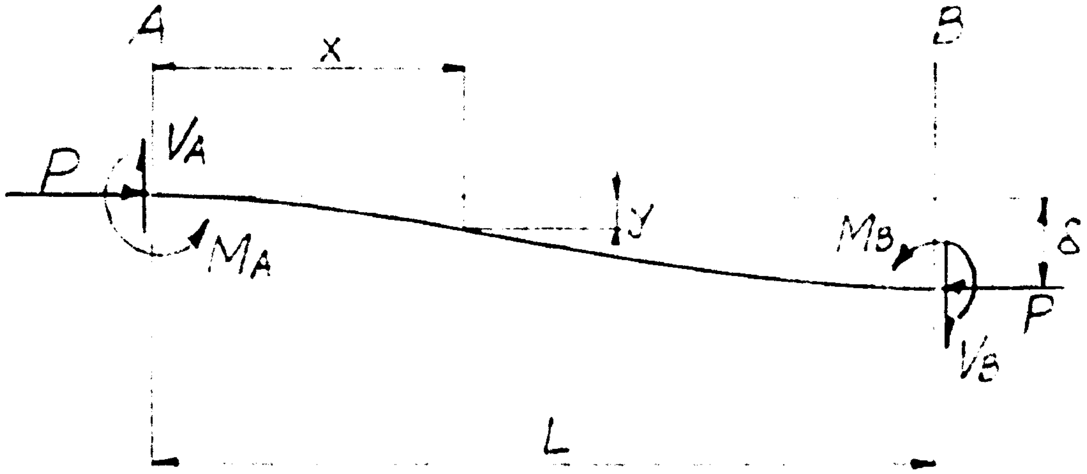


Figure 2. Compression member with its right end deflected.

in which $V_A = \frac{M_A + M_B - P\delta}{L}$ and from symmetry $M_A = M_B$. Letting $M = M_A = M_B$ and substituting k^2 for $\frac{P}{EI}$, the solution of equation (6) is

$$y = C_1 \operatorname{Cos} kx + C_2 \operatorname{Sin} kx + \frac{M}{P} - \frac{2M}{PL} x + \frac{\delta x}{L} \tag{7}$$

The constants C_1 and C_2 in equation (7) are determined from the boundary conditions, $x = 0$, $y = 0$, and $x = L$, $y = \delta$ resulting in

$$C_1 = -\frac{M}{P} \quad \text{and} \quad C_2 = \frac{M}{P} \frac{1 + \operatorname{Cos} kL}{\operatorname{Sin} kL}$$

Therefore,

$$y = -\frac{M}{P} \operatorname{Cos} kx + \frac{M}{P} \frac{1 + \operatorname{Cos} kL}{\operatorname{Sin} kL} \operatorname{Sin} kx + \frac{M}{P} - \frac{2M}{PL} x + \frac{\delta x}{L} \tag{8}$$

Taking the derivative of equation (8) with respect to x, and introducing the boundary condition, $x = 0$, $\frac{dy}{dx} = 0$ results in

$$M = \frac{EI \qquad \delta}{L^2(\phi_n - \phi_f)} \tag{9}$$

Writing equation (9) in terms of C and S from equations (4) and (5) results in

$$M = \frac{S(1 + C)\delta}{L} \tag{10}$$

The value of the shear V_A or V_B is equal to the translational stiffness of the member when $\delta = 1$

$$T = V_A = V_B = \frac{2M - P}{L}$$

$$= \frac{2S(1 + C)}{L^2} - \frac{P}{L}$$

Substituting S''' for S(1 + C) results in

$$T = \frac{2S'''}{L^2} - \frac{P}{L} \tag{11}$$

All of the previous derivations were made for members in which the axial force was compression. When the axial force is tension some changes are brought about in the equations. The elastic constants S, C, and T will, therefore, be derived for a member with a tension axial force.

The differential equation for the deflection of the member in Figure 3 is

$$EI \frac{d^2y}{dx^2} = - M_A + V_A x + Py \tag{12}$$

in which $V_A = \frac{M_A + M_B}{L}$. Substituting k^2 for $\frac{P}{EI}$, the solution of equation (12) is

$$y = C_1 \text{ Cosh } kx + C_2 \text{ Sinh } kx + \frac{M_A}{P} - \frac{M_A + M_B}{PL}\ x \tag{13}$$

The constants C_1 and C_2 in equation (13) are determined from the boundary conditions, x = 0, y = 0, and x = L, y = 0 resulting in

$$C_1 = - \frac{M_A}{P} \quad \text{and } C_2 = \frac{1}{P \text{ Sinh } kL} (M_A \text{ Cosh } kL + M_B)$$

Therefore,

$$y = - \frac{M_A}{P} \text{ Cosh } kx + \frac{M_A \text{ Cosh } kL + M_B}{P \text{ Sinh } kL} \text{ Sinh } kx + \frac{M_A}{P} - \frac{M_A + M_B}{PL}\ x \tag{14}$$

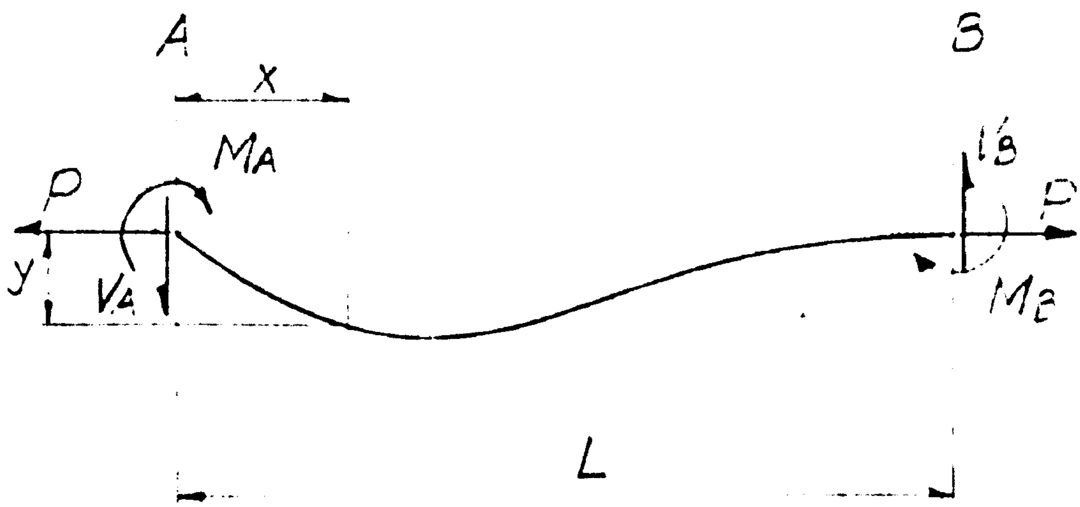


Figure 3. Tension member with its left end rotated.

Taking the derivative of equation (14) with respect to x, introducing the boundary condition $x = L$, $\frac{dy}{dx} = 0$, and then making the substitutions

$$\phi'_f = -\frac{1}{(kL)^2} \quad (kL \text{ Csch } kL - 1)$$

$$\phi'_n = -\frac{1}{(kL)^2} \quad (1 - kL \text{ Coth } kL)$$

the carry-over factor is found to be

$$C = \frac{M_B}{M_A} = \frac{\phi'_f}{\phi'_n} \qquad (15)$$

The stiffness factor is found with the boundary condition $x = 0$, $\frac{dy}{dx} = 1$

$$S = M_A = \frac{EI}{L} \quad \frac{\phi'_n}{(\phi'_n)^2 - (\phi'_f)^2} \qquad (16)$$

As for members with axial compression, values of C and $\frac{S}{\frac{4EI}{L}}$ for members with axial tension have been tabulated (6) (7) for various values of kL. Approximate values of C and S can be obtained from the curves included in Appendix A.

The expression for the translational stiffness of the member also depends on the type of axial force.

The differential equation for the deflection of the member in Figure 4 is

$$EI \frac{d^2y}{dx^2} = M_A - V_A x + Py \qquad (17)$$

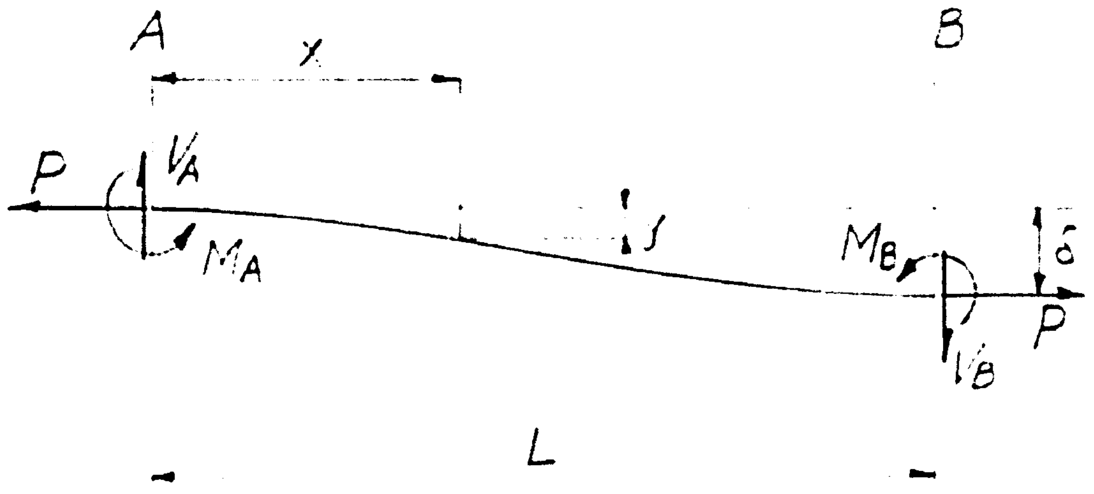


Figure 4. Tension member with its right end deflected.

in which $V_A = \frac{M_A + M_B + P\delta}{L}$ and from symmetry $M_A = M_B$.

Letting $M = M_A = M_B$ and substituting k^2 for $\frac{P}{EI}$, the solution of equation (17) is

$$y = C_1 \text{ Cosh } kx + C_2 \text{ Sinh } kx - \frac{M}{P} + \frac{2M + P\delta}{PL} x \qquad (18)$$

The constants C_1 and C_2 in equation (18) are determined from the boundary conditions, $x = 0$, $y = 0$, and $x = L$, $y = \delta$ resulting in

$$C_1 = \frac{M}{P} \quad \text{and} \quad C_2 = -\frac{M}{P} \frac{1 + \text{Cosh } kL}{\text{Sinh } kL}$$

Therefore,

$$y = \frac{M}{P} \text{ Cosh } kx - \frac{M}{P} \frac{1 + \text{Cosh } kL}{\text{Sinh } kL} \text{ Sinh } kx - \frac{M}{P} + \frac{2M + P\delta}{PL} x \qquad (19)$$

Taking the derivative of equation (19) with respect to x and introducing the boundary condition $x = 0$, $\frac{dy}{dx} = 0$ results in

$$M = \frac{EI\,\delta}{L^2(\phi_n' - \phi_f')} \qquad (20)$$

Writing equation (20) in terms of C and S from equations (15) and (16) results in

$$M = \frac{S(1 + C)\delta}{L} \qquad (21)$$

Equation (21) for a member with axial tension has exactly the same form as equation (10) for a member with axial compression. Even though the form of these equations is the same, there is a difference in the individual values for C and S depending upon whether the axial load is tension or compression.

The value of the shear V_A or V_B is the translational stiffness for the member when $\delta = 1$

$$T = V_A = V_B = \frac{2M + P}{L}$$

$$= \frac{2S(1 + C)}{L^2} + \frac{P}{L}$$

Substituting S''' for $S(1 + C)$

$$T = \frac{2S'''}{L^2} + \frac{P}{L} \qquad (22)$$

In later derivations it was desirable to have a general equation for the moments at the ends of a member for the general conditions of displacement and rotation of both ends of the member as is indicated in Figure 5.

The differential equation for the deflection of the member in Figure 5 is

$$EI \frac{d^2y}{dx^2} = - M_A + V_A x - P(y - y_A) \qquad (23)$$

in which $V_A = \frac{M_A + M_B + P(y_B - y_A)}{L}$. Substituting k^2 for $\frac{P}{EI}$, the solution of equation (23) is

$$y = C_1 \operatorname{Cos} kx + C_2 \operatorname{Sin} kx - \frac{M_A}{P} + \frac{M_A + M_B + P(y_B - y_A)}{PL} x$$

$$+ y_A \qquad (24)$$

The constants C_1 and C_2 in equation (24) are determined from the boundary conditions, $x = 0$, $y = y_A$, and $x = L$, $y = y_B$ resulting in

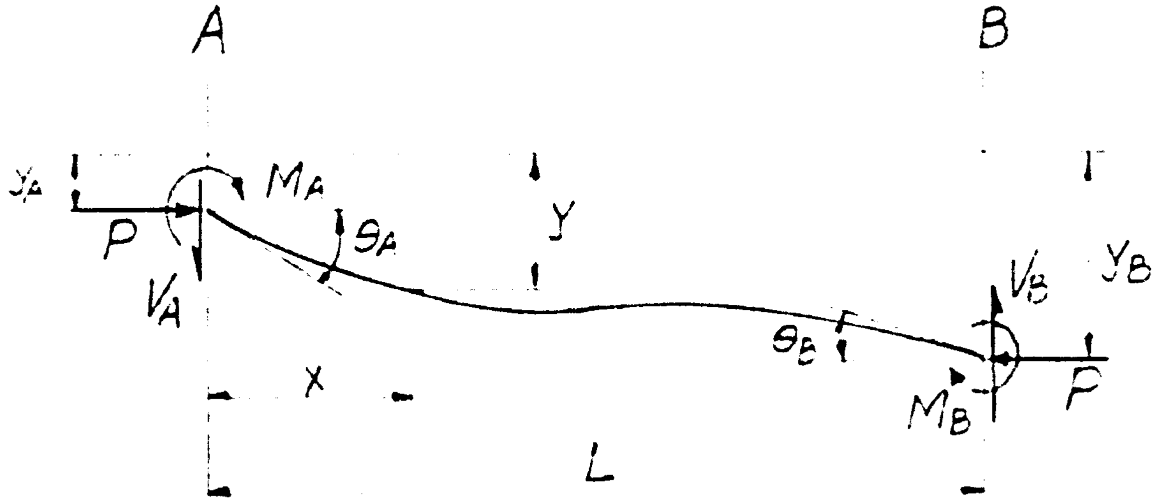


Figure 5. Compression member with both ends rotated and deflected.

$$C_1 = \frac{M_A}{P} \quad \text{and} \quad C_2 = -\frac{M_A \cos kL + M_B}{P \sin kL}$$

Therefore,

$$y = \frac{M_A}{P} \cos kx - \frac{M_A \cos kL + M_B}{P \sin kL} \sin kx - \frac{M_A}{P} +$$

$$\frac{M_A + M_B + P(y_B - y_A)}{PL} x + y_A \qquad (25)$$

Taking the derivative of equation (25) with respect to x and introducing the boundary conditions, $x = 0$, $\frac{dy}{dx} = \theta_A$ and $x = L$, $\frac{dy}{dx} = \theta_B$, two equations are obtained that can be solved simultaneously for M_A and M_B:

$$M_A = \frac{\phi_n}{\phi_n^2 - \phi_f^2} \frac{EI}{L}\left[\theta_A + \theta_B \frac{\phi_f}{\phi_n} - \left(1 + \frac{\phi_f}{\phi_n}\right)\left(\frac{y_B - y_A}{L}\right)\right] \qquad (26)$$

$$M_B = \frac{\phi_n}{\phi_n^2 - \phi_f^2} \frac{EI}{L}\left[\theta_B + \theta_A \frac{\phi_f}{\phi_n} - \left(1 + \frac{\phi_f}{\phi_n}\right)\left(\frac{y_B - y_A}{L}\right)\right] \qquad (27)$$

Writing equations (26) and (27) in terms of C and S from equations (4) and (5) results in

$$M_A = S\left[\theta_A + \theta_B C - (1 + C)\left(\frac{y_B - y_A}{L}\right)\right] \qquad (28)$$

$$M_B = S\left[\theta_B + \theta_A C - (1 + C)\left(\frac{y_B - y_A}{L}\right)\right] \qquad (29)$$

Equations (28) and (29) are derived for a compression axial load; the form of the equations for a tension axial load is exactly the same. Even though the form of these equations remains the same, there is a difference in the individual values for S and C depending upon whether the axial load is tension or compression.

For the beam-column, Figure 6(a), the procedure and factors that are required for a moment distribution solution for the moments due to an externally applied moment at one of the joints are as follows:

Apply an external moment at joint C. The structure will deflect, in general, as shown in Figure 6(b). The joint C is permitted to rotate but all of the other joints are held against rotation with a temporary externally applied moment.

The external moment applied at C must be distributed between the ends of the two members that meet at joint C. Stiffness factors will be derived which will be used to determine the distribution factors at C.

A first type of carry-over factor for the member CD must be determined so that the moment M_{DC} can be computed as this carry-over factor multiplied by the balancing moment, M_{CD}, at joint C.

A second type of carry-over factor must be determined so that the moments M_{DE} and M_{ED} can be found as this carry-over factor multiplied by the moment M_{CD} at joint C.

A third type of carry-over factor must be determined so that the moments M_{EF} and M_{FE} can be found as this carry-over factor multiplied by the moment M_{DE}.

The third type of carry-over factor is used to determine the moments induced in any additional members to the right.

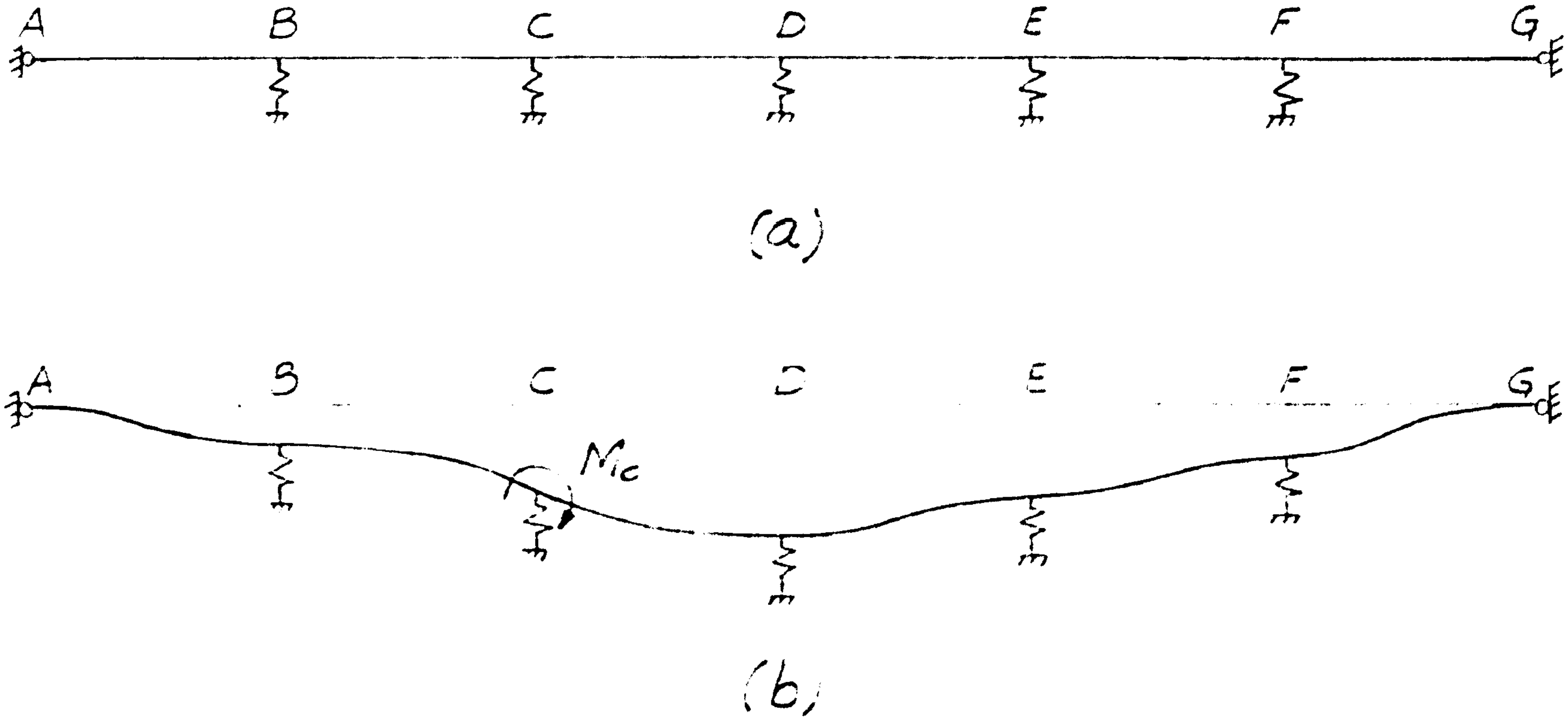


Figure 6. Elastically supported beam-column with Joint C rotated.

The same procedure as described above is used to determine the induced moments at the ends of the members to the left of joint C. This completes one cycle of balancing one joint and determining the induced moments in the other members. The other joints will be balanced in a similar manner, one at a time, and moments carried over until all the joints are balanced.

The elastic constants of the members, stiffness and carry-over factors, are the same for each end of the member when there is no relative displacement of the ends of the member. This is true for a member of constant cross section throughout its length when axial forces are considered as well as when axial forces are non-existent or neglected. However, when the ends of the members are elastically restrained, relative displacement of the ends will occur, and the values for the stiffness and carry-over factors must be determined for each end of the member. The magnitude of these factors will depend, therefore, on the magnitude of the stiffness of the elastic supports, as well as all of the other factors that were included in the previous derivations made for the expressions for stiffness and carry-over factors when no relative displacement of the ends of the member occurred.

Figure 7(a) shows a segment of any two consecutive

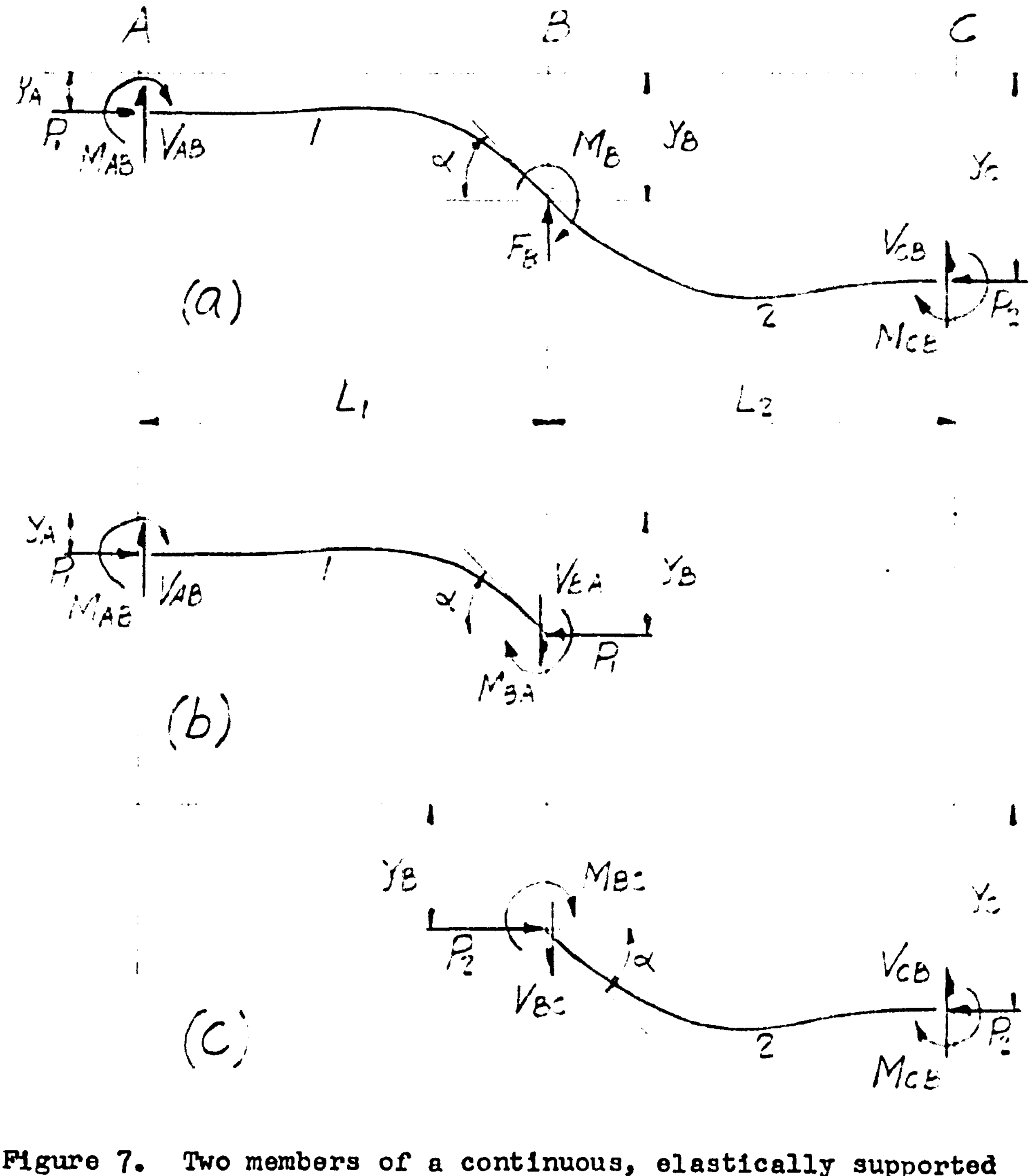


Figure 7. Two members of a continuous, elastically supported beam-column with Joint B rotated, and with Joints A, B, and C deflected.

members of the beam-column shown in Figure 6(a). In order to simplify the notation for generalized equations which are to be derived, the members of any segment of any two consecutive members will be referred to as members 1 and 2, and the joints as A, B, and C. This notation will be followed in all subsequent derivations.

Apply an external moment M_B at joint B producing a rotation of α radians. All of the joints are free to deflect subject to their elastic supports, joint B is permitted to rotate under the applied moment, and all of the other joints are fixed against rotation.

The effect of the elastic support at A and the entire structure to the left is represented by V_{AB} which is the shear in the member AB at a point just to the right of the elastic support at A. The shear V_{AB} will be expressed as $t'_{1L}y_A$ in which y_A is the deflection of joint A and t'_{1L} is defined as the force per unit of deflection for the left end of member 1 and the entire structure to the left when no joint is permitted to rotate. Likewise V_{CB}, the shear just to the left of the elastic support at C, represents the effect of the elastic support at C and the entire structure to the right. The shear V_{CB} will be expressed as $t'_{2R}y_C$ in which y_C is the deflection of joint C and t'_{2R} is defined as the force per unit of deflection for the right end of member 2 and the entire structure to the right when no joint is permitted to rotate. The external force F_B at joint B can be expressed as $t_B y_B$ in which y_B is the deflection of joint B and t_B is defined as the force per unit of displacement for the elastic support at B.

Referring to Figures 7(b) and (c) and to equations (28) and (29) the following equations for the moments at both ends of the two members can be written:

$$M_{AB} = S_1 \left[C_1 \alpha - R_{1R}(1 + C_1) \right] \tag{30}$$

$$M_{BA} = S_1 \left[\alpha - R_{1R}(1 + C_1) \right] \tag{31}$$

$$M_{BC} = S_2 \left[\alpha - R_{2L}(1 + C_2) \right] \tag{32}$$

$$M_{CB} = S_2 \left[C_2 \alpha - R_{2L}(1 + C_2) \right] \tag{33}$$

in which R_{1R} is defined as $\dfrac{y_B - y_A}{L_1}$, and R_{2L} is defined as $\dfrac{y_C - y_B}{L_2}$.

The relative magnitude of the moments M_{BA} and M_{BC} will give the necessary information to distribute the moment applied at joint B to the two members coming into joint B. From an inspection of the above equations it appears that these moments may not vary linearly with the angle of rotation of joint B. Therefore, the derivation will be made for an angle of α radians instead of an angle of one radian as is usually used in the derivation of stiffness factors.

To obtain the values for the moments, an expression of

R_{1R} and R_{2L} will be required.

Referring to Figure 7(b) and summing moments about B:

$$M_{AB} + M_{BA} + P_1(y_B - y_A) + t'_{1L}y_AL_1 = 0 \tag{34}$$

Referring to Figure 7(c) and summing moments about B:

$$M_{BC} + M_{CB} + P_2(y_C - y_B) - t'_{2R}y_CL_2 = 0 \tag{35}$$

Referring to Figure 7(a) and summing forces in the y direction:

$$t'_{1L}y_A + t_By_B + t'_{2R}y_C = 0 \tag{36}$$

Substituting equations (30) and (31) into equation (34) and making a substitution using equation (11) results in

$$S_1'''\alpha - T_1L_1^2 \frac{y_B - y_A}{L_1} + t'_{1L}y_AL_1 = 0 \tag{37}$$

Substituting equations (32) and (33) in equation (35) and making a substitution using equation (11) results in

$$S_2'''\alpha - T_2L_2^2 \frac{y_C - y_B}{L_2} - t'_{2R}y_CL_2 = 0 \tag{38}$$

Equations (37) and (38) are derived for a compression axial load; the form of the equations for a tension axial load is exactly the same. Even though the form of these equations remains the same, there is a difference in the individual values for S''' and T depending upon whether the axial load is tension or compression.

The three equations (36), (37), and (38) are solved simultaneously for y_A, y_B, and y_C. These quantities are used to solve for the terms R_{1R} and R_{2L} required in equations (30), (31), (32), and (33).

$$R_{1R} = \frac{y_B - y_A}{L_1} = \left[\frac{S_1''' \, (t'_{1R} + t'_{1L}) - S_2''' \, \frac{L_1}{L_2} \frac{T'_{2L}}{T_2} \, t'_{1L}}{L_1^2 (t'_{1L} + T_1)(t'_{1R} + T'_{1R})}\right] \alpha \quad (39)$$

$$R_{2L} = \frac{y_C - y_B}{L_2} = \left[\frac{S_2''' \, (t'_{2L} + t'_{2R}) - S_1''' \, \frac{L_2}{L_1} \frac{T'_{1R}}{T_1} \, t'_{2R}}{L_2^2 (t'_{2R} + T_2)(t'_{2L} + T'_{2L})}\right] \alpha \quad (40)$$

in which T'_{1R} is defined as the force per unit of deflection for the right end of member 1 and the entire structure to the left when no joint is permitted to rotate, but is free to deflect subject to the elastic supports. Likewise T'_{2L} is defined as the force per unit of deflection for the left end of member 2 and the entire structure to the right.

Also, the equations relating the effect of elastic supports in series and in parallel have been used to reduce equations (39) and (40) to the form given above. Examples of these equations are

$t'_{2L} = t_B + T'_{1R}$ for elastic supports in series, and

$T'_{1R} = \dfrac{t'_{1L} T_1}{t'_{1L} + T_1}$ for elastic supports in parallel.

It is noted from equations (39) and (40) that the values of R_{1R} and R_{2L} vary linearly with α. If these values are substituted in equations (30), (31), (32), and (33), it is obvious that the moments at the ends of the member in Figure

7 also vary linearly with α. Therefore, the usual definition based on one radian of rotation can be used for the stiffness factor.

When, however, the ends of the members are elastically restrained, relative displacement of the ends will occur, and the value for the stiffness factor must be determined for each end of the member. The symbol S_{1R} will be used to denote this stiffness factor for the right end of member 1; it will be defined as the moment applied at the right end of member 1 with the magnitude necessary to produce a rotation of one radian at the right end when the ends of the member are elastically restrained against deflection and the left end of the member is not permitted to rotate. Also, from the definition of the quantities that have been used in the derivation, all of the joints to the left of the member are free to deflect subject to the elastic supports but none of these joints is permitted to rotate. The values of S_{1R} and S_{2L} are, therefore, M_{BA} and M_{BC} when α equals one radian

$$S_{1R} = M_{BA} = S_1 \left[1 - R_{1R}(1 + C_1) \right] \tag{41}$$

$$S_{2L} = M_{BC} = S_2 \left[1 - R_{2L}(1 + C_2) \right] \tag{42}$$

With the above information the first type of carry-over factor can be found. This carry-over factor is used to find the moment induced at the far end of the members that meet at the joint that is being balanced. As an example, the symbol C_{a1R} is used to denote this carry-over factor. The subscripts a1R indicate that the carry-over factor is for the right end of member 1, that the right end of member 1 has been balanced, and that the balancing moment at the right end of member 1 multiplied by the carry-over factor C_{a1R} gives the induced moment at the left end of member 1. For any value of α the carry-over factors will be independent of α and are determined for the members in Figure 7(a) by using equations (30), (31), (32), and (33)

$$C_{a1R} = \frac{M_{AB}}{M_{BA}} = \frac{C_1 - R_{1R}(1 + C_1)}{1 - R_{1R}(1 + C_1)} \tag{43}$$

and

$$C_{a2L} = \frac{M_{CB}}{M_{BC}} = \frac{C_2 - R_{2L}(1 + C_2)}{1 - R_{2L}(1 + C_2)} \tag{44}$$

A special formula for R_{1R} will have to be derived for the situation when A is the left end of the entire beam-column as shown in Figure 8(a). An external moment is applied at joint B, which is permitted to rotate and deflect. Joint A is not permitted to rotate or deflect. All of the joints to the right of B are permitted to deflect but are held rigidly against rotation.

Referring to Figures 8(b) and (c) and to equations (28) and (29), the following equations for the moments at both ends of the two members can be written:

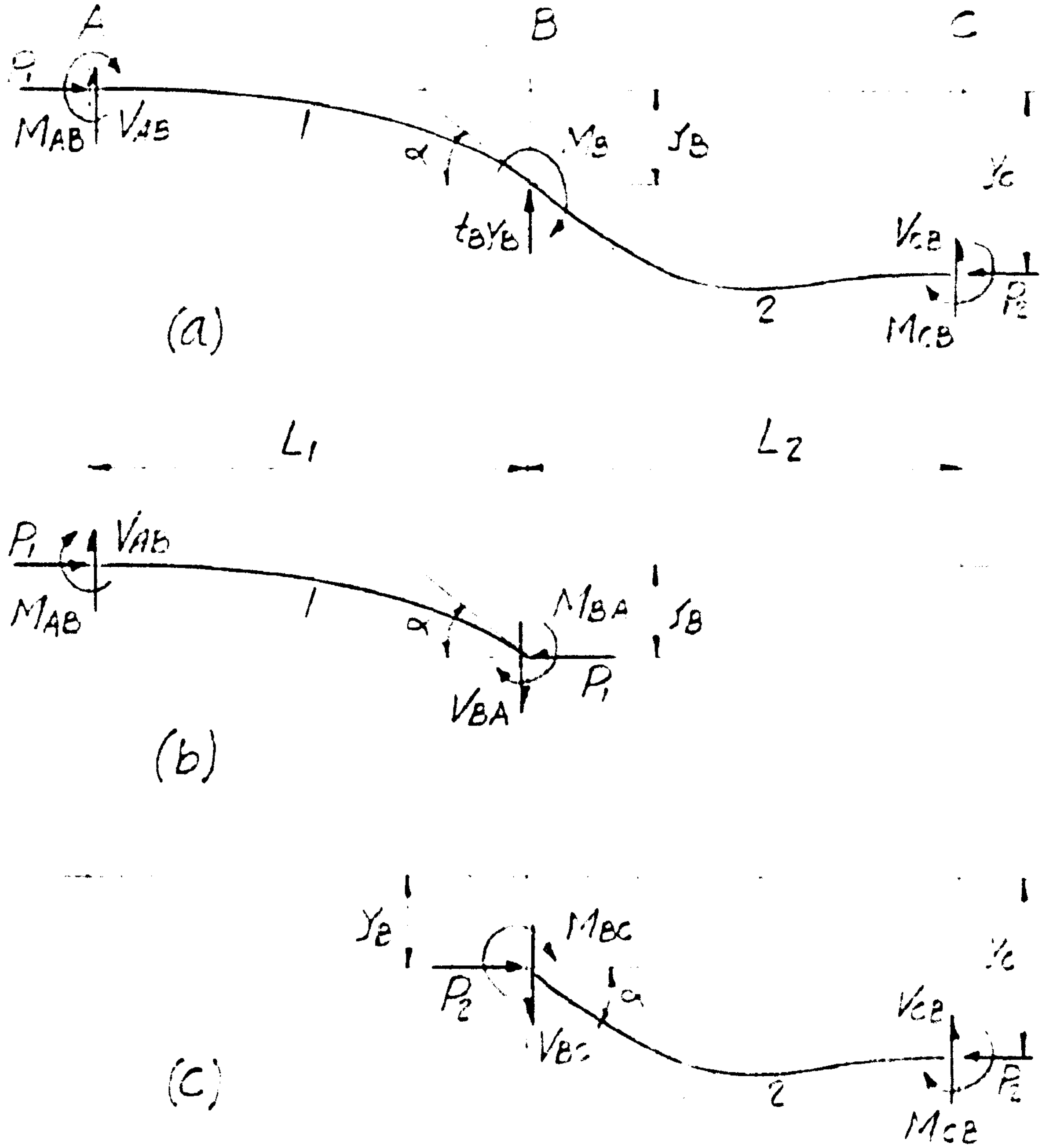


Figure 8. Two members of a continuous, elastically supported beam-column with Joint B rotated, and with Joints B and C deflected.

$$M_{AB} = S_1 \left[C_1\alpha - R_{1R}(1 + C_1) \right] \tag{45}$$

$$M_{BA} = S_1 \left[\alpha - R_{1R}(1 + C_1) \right] \tag{46}$$

$$M_{BC} = S_2 \left[\alpha - R_{2L}(1 + C_2) \right] \tag{47}$$

$$M_{CB} = S_2 \left[C_2\alpha - R_{2L}(1 + C_2) \right] \tag{48}$$

Referring to Figure 8(b) and summing moments about B:

$$M_{AB} + M_{BA} + P_1 y_B + V_{AB} L_1 = 0 \tag{49}$$

Referring to Figure 8(c) and summing moments about B:

$$M_{BC} + M_{CB} + P_2(y_C - y_B) - V_{CB} L_2 = 0 \tag{50}$$

By definition $V_{CB} = t'_{2R} y_C$ (51)

Referring to Figure 8(a) and summing forces in the y direction:

$$V_{AB} + t_B y_B + t'_{2R} y_C = 0 \tag{52}$$

Substituting equations (45), (46), and (52) in equation (49) and making a substitution using equation (11) results in

$$\frac{S_1'''\alpha}{L_1} - T_1 y_B - t_B y_B - t'_{2R} y_C = 0 \tag{53}$$

Substituting equations (47), (48), and (51) in equation (50) and making a substitution using equation (11) results in

$$\frac{S_2'''\alpha}{L_2} - T_2(y_C - y_B) - t'_{2R} y_C = 0 \tag{54}$$

Equations (53) and (54) are solved simultaneously for y_B and y_C. Then,

$$R_{1R} = \frac{y_B}{L_1} = \left[\frac{S_1''' - S_2''' \dfrac{T'_{2L}}{T_2} \dfrac{L_1}{L_2}}{L_1^2 \, (T_1 + t'_{1R})} \right] \alpha \tag{55}$$

A second special formula for R_{1L} will be derived for the situation when A is the left end of the entire beam-column as shown in Figure 9(a). An external moment is applied at joint A which is permitted to rotate but not deflect. All of the joints to the right of A are permitted to deflect but are held rigidly against rotation.

Referring to Figure 9(b) and to equations (28) and (29) the following equations for the moments at both ends of the member can be written:

$$M_{AB} = S_1 \left[\alpha - R_{1L}(1 + C_1) \right] \tag{56}$$

$$M_{BA} = S_1 \left[\alpha C_1 - R_{1L}(1 + C_1) \right] \tag{57}$$

Referring to Figure 9(b) and summing moments about A:

$$M_{AB} + M_{BA} + P_1 y_B - V_{BA} L_1 = 0 \tag{58}$$

By definition,

$$V_{BA} = t'_{1R} y_B \tag{59}$$

Substituting equations (56), (57), and (59) in equation (58) and making a substitution using equation (11) results in

$$\frac{S_1''' \alpha}{L_1} - T_1 y_B - t'_{1R} y_B = 0 \tag{60}$$

Therefore,

$$R_{1L} = \frac{y_B}{L_1} = \frac{S_1'''}{L_1^2 (T_1 + t'_{1R})} \alpha \tag{61}$$

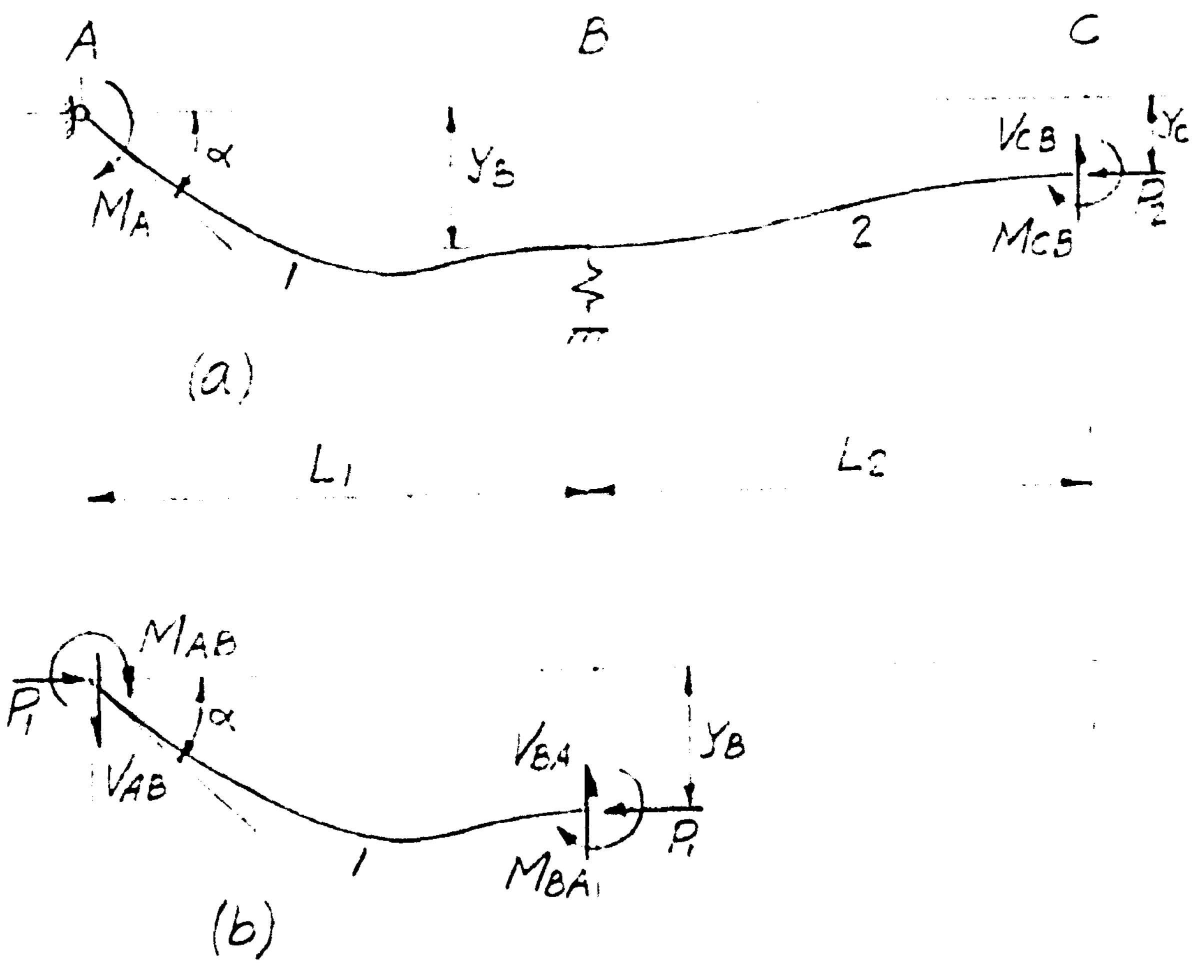


Figure 9. Two members of a continuous, elastically supported beam-column with Joint A rotated, and with Joints B and C deflected.

The second type of carry-over factor will be derived for the conditions shown in Figure 10. When the joint at C is balanced, moments are introduced at the ends of member 1 as is indicated in Figures 10(a) and (b). All of the joints of the beam-column except C are prevented from rotating but the joints are free to deflect subject to the elastic restraints of the supports. The carry-over factor will be denoted as C_{b2R}. The subscripts b2R indicate that the carry-over factor is for the right end of member 2, that the right end of member 2 has been balanced, and that the carry-over factor C_{b2R} multiplied by the balancing moment at the right end of member 2 gives the induced moments at both ends of the first member to the left of member 2.

Referring to figures 10(b) and (c) and to equations (28) and (29) the following equations for the moments at both ends of the two members can be written:

$$M_{AB} = M_{BA} = S_1 R_1 (1 + C_1) \qquad (62)$$

$$M_{BC} = S_2 \left[C_2 \alpha - R_{2R}(1 + C_2) \right] \qquad (63)$$

$$M_{CB} = S_2 \left[\alpha - R_{2R}(1 + C_2) \right] \qquad (64)$$

in which R_1 is defined as $\frac{y_B - y_A}{L_1}$.

Referring to Figure 10(b) and summing moments about B:

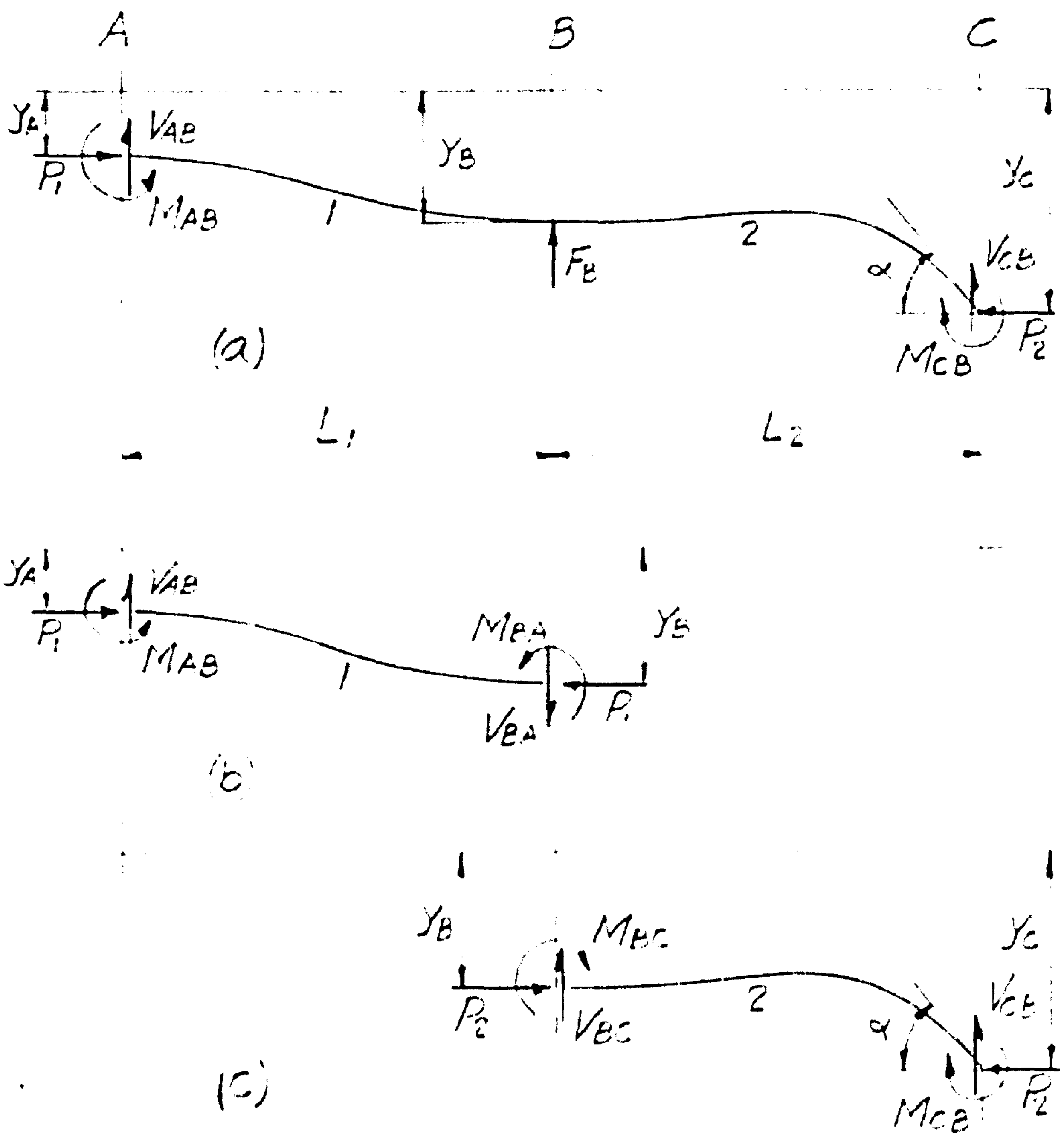


Figure 10. Two members of a continuous, elastically supported beam-column with Joint C rotated and with Joints A, B, and C deflected.

$$M_{AB} + M_{BA} - P_1(y_B - y_A) - V_{AB}L_1 = 0 \qquad (65)$$

By definition,

$$V_{AB} = t'_{1L}y_A \qquad (66)$$

Referring to Figure 10(c) and summing moments about C:

$$M_{BC} + M_{CB} + P_2(y_C - y_B) + V_{BC}L_2 = 0 \qquad (67)$$

By definition,

$$V_{BC} = t'_{2L}y_B \qquad (68)$$

An equation summing forces in the y direction can be written but it will not produce an equation independent of equation (65).

Substituting equations (62) and (66) in equation (65) and making a substitution using equation (11) results in

$$T_1(y_B - y_A) - t'_{1L}y_A = 0 \qquad (69)$$

Substituting equations (63), (64), and (68) in equation (67) and making a substitution using equation (11) results in

$$\frac{S_2'''\alpha}{L_2} - T_2(y_C - y_B) + t'_{2L}y_B = 0 \qquad (70)$$

The equations (69) and (70) are solved simultaneously for y_A and y_B in terms of R_{2R}.

$$R_{2R} = \frac{y_C - y_B}{L_2}$$

Then,

$$R_1 = \frac{y_B - y_A}{L_1} = \frac{L_2}{L_1}\,\frac{T'_{1R}}{T_1 t'_{2L}}\left(T_2 R_{2R} - \frac{S_2''' \alpha}{L_2^2}\right) \qquad (71)$$

The carry-over factor denoted by C_{b2R} is expressed as

$$C_{b2R} = \frac{-M_{AB}}{M_{BC}} = \frac{-M_{BA}}{M_{BC}} = \frac{-S_1 R_1 (1 + C_1)}{S_2\left[\alpha - R_{2R}(1 + C_2)\right]} \qquad (72)$$

The negative sign is introduced on M_{AB} and M_{BA} because all of the moments are expressed as positive when they are clockwise on a free-body of the member; M_{AB} and M_{BA} are both shown as counterclockwise moments in Figure 10(b), therefore the negative signs.

The numerator and denominator of equation (72) vary linearly with α and, therefore, the carry-over factor C_{b2R} is independent of α. The denominator has been defined as the stiffness factor S_{2R} for an α equal to one radian. Therefore, the equation (72) can be simplified for the case of α equal to one radian as

$$C_{b2R} = \frac{S_1'''}{S_{2R}}\,\frac{L_2}{L_1}\,\frac{T'_{1R}}{T_1 t'_{2L}}\left(\frac{S_2'''}{L_2^2} - T_2 R_{2R}\right) \qquad (73)$$

A third type of carry-over factor will be derived using Figure 11. The moments at the ends of member 2 have been determined with the previous carry-over factor. Member 2 in Figure 11 corresponds to member 1 in Figure 10. The joints A, B, and C are held against rotation but are free to deflect subject to the elastic restraints of the supports. The

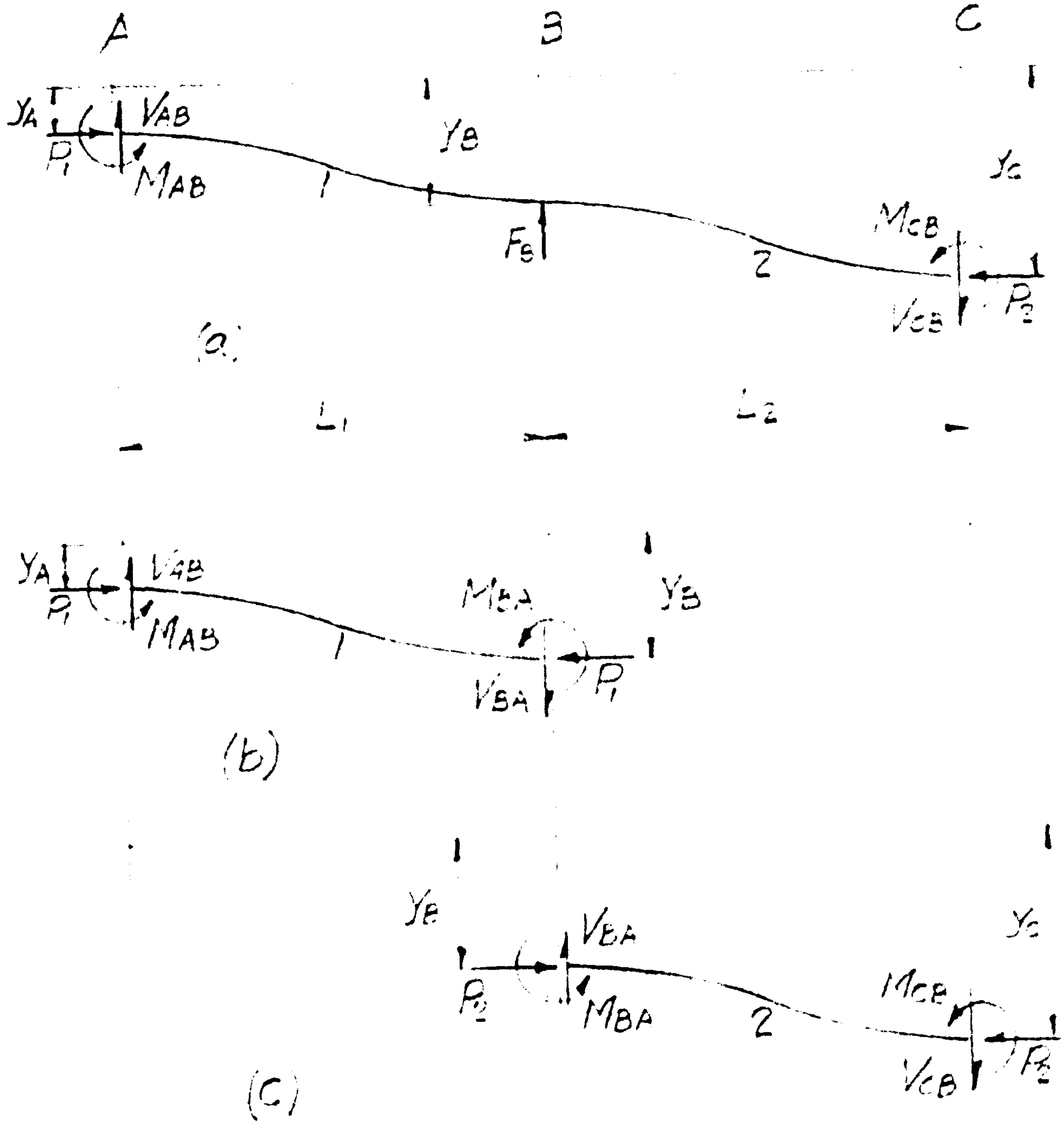


Figure 11. Two members of a continuous, elastically supported beam-column with Joints A, B, and C deflected by the rotation of a joint to the right of C.

deflections are caused from a moment applied at a joint to the right of joint C. The carry-over factor will be denoted as C_{c2R}. The subscripts c2R indicate that the carry-over factor is for the right end of member 2, that a joint to the right of the right end of member 2 has been balanced, and that the carry-over factor C_{c2R} multiplied by the induced moment at the right end of member 2 gives the induced moments at both ends of the first member to the left of member 2.

Referring to Figures 11(b) and (c) and to equations (28) and (29) the following equations for the moments at both ends of the two members can be written:

$$M_{AB} = M_{BA} = S_1R_1(1 + C_1) \tag{74}$$

$$M_{BC} = M_{CB} = S_2R_2(1 + C_2) \tag{75}$$

Referring to Figure 11(c) and summing moments about C:

$$2M_{BC} - P_2(y_C - y_B) - V_{BA}L_2 = 0 \tag{76}$$

By definition,

$$V_{BA} = t'_{2L}y_B \tag{77}$$

From equation (75),

$$y_C - y_B = \frac{M_{BC}L_2}{S_2'''} \tag{78}$$

Substituting equations (77) and (78) in equation (76) and making a substitution using equation (11) results in

$$y_B = \frac{M_{BC}T_2L_2}{t'_{2L}S_2'''} \tag{79}$$

Referring to Figure 11(b) and summing moments about B:

$$2M_{AB} - P_2(y_B - y_A) - V_{AB}L_1 = 0 \qquad (80)$$

By definition,

$$V_{AB} = t'_{1L}y_A \qquad (81)$$

Substituting equations (74) and (81) in equation (80) and making a substitution using equation (11) results in

$$y_A = y_B \frac{T'_{1R}}{t'_{1L}} \qquad (82)$$

From equations (79) and (82),

$$R_1 = \frac{y_B - y_A}{L_1} = \frac{M_{BC}}{S_2'''} \frac{L_2}{L_1} \frac{T_2}{t'_{2L}} \frac{T'_{1R}}{T_1} \qquad (83)$$

The carry-over factor denoted by C_{o2R} is expressed as

$$C_{o2R} = \frac{-M_{AB}}{-M_{CB}} = \frac{S_1'''}{S_2'''} \frac{L_2}{L_1} \frac{T_2}{t'_{2L}} \frac{T'_{1R}}{T_1} \qquad (84)$$

Equation (84) is used to find the induced moments in any members to the left of member 1.

The equations that are derived in this section for use in applying the series criterion are used as follows:

1. Determine the carry-over factor C by using equation (4) or (15).
2. Determine the stiffness factor S by using equation (5) or (16).
3. Determine the translational stiffness T by using equation (11) or (22).

4. Determine the quantities R_{1R} and R_{2L} at each joint by using the appropriate equation (39), (40), (55), or (61).
5. Determine the stiffness factors S_{1R} and S_{2L} at each joint using equations (41) and (42).
6. Determine the distribution factors, D.F., at the joints on the basis of the relative magnitudes of the stiffness factors for the ends of the members meeting at the joint.
7. Determine the carry-over factors C_a, C_b, and C_c using the appropriate equation (43), (44), (73), or (84).

A short numerical example is included in Appendix B to demonstrate the use of the series criterion.

Stiffness Criterion

The stiffness criterion is another one of the criteria used to check the stability in a structure. The stiffness criterion states that at the critical buckling load the rotational and translational stiffnesses of every joint in the structure are zero. This statement must be qualified, however, since the joint in question may be at a node or at a point of maximum deflection on the buckling curve of the beam-column. If the joint is at a node, the translational stiffness may not be zero and if the joint is at a point of maximum deflection, the rotational stiffness may not be zero.

In applying the stiffness criterion a procedure of trial must be used; that is, the rotational and translational stiffnesses must be computed for an assumed stress situation and, in general, the stiffnesses computed may be different than zero. Therefore, more generalized statements for the stiffness criterion are:

1. When an external clockwise moment, under a condition of equilibrium, is required at a joint to rotate the joint in a clockwise direction, there is an indication of stability; but, when the external moment necessary to hold the joint in equilibrium is opposite to the direction of rotation, then there is an indication of instability.
2. When a lateral force, under a condition of equilibrium, is applied at a joint and when the direction of the deflection of the joint is in the direction of the applied force, there is an indication of stability; but, when, under a condition of equilibrium, the direction of the force must be opposite to the direction of the deflection of the joint, then a condition of instability is indicated.

The first of the two statements is essentially the same as the series criterion.

The rotational and translational stiffnesses are determined, for example, for the right end of a member considering the left end to be elastically restrained against deflection and rotation by the members and supports to the left of the member that is being considered. The rotational and translational stiffnesses of a member are the stiffnesses of one end of the member, considering the member and the entire structure beyond its far end, while the stiffnesses of a joint are the stiffnesses at the joint considering the entire structure on both sides of the joint. The elastic restraints supplied to the left end of the member by the members and supports to the left are expressed in terms of the rotational and translational stiffnesses of the right end of the member immediately to the left. Therefore, the procedure used is to start at the end of the structure and work toward the joint by a set of computations for the right end of a member based on the stiffnesses at the left which have been previously computed for the right end of the member to the left.

The stiffness criterion as developed in this thesis was applied to a beam-column that has a joint about which the structure is symmetrical. The symmetry is required for the physical structure and also the stress situation in the members. It was also assumed that the ends of the beam-column are hinged. Under these conditions the joint about which the structure is symmetrical will be either a node or a point of maximum deflection on the buckling curve. The procedure was developed on this basis since the three-span pony truss used as an example satisfied the above conditions.

Derivation of Formulas for the Stiffness Criterion

Some additional symbols will have to be defined. For the following definitions the structure has been cut just to the right of joint B and only that part of the structure to the left is considered. The rotational stiffness, S_{Bzz}, of a member is the moment per unit of rotation necessary to rotate joint B when joint B is not permitted to deflect, and all of the joints to the left of B are free to deflect and rotate subject to the elastic restraints of the members and supports; in this case, S_{Byz} is the force per unit of rotation necessary at B to prevent any deflection of joint B when it is rotated. The translational stiffness, S_{Byy}, of a member is the force per unit of deflection necessary at B to deflect joint B when B is not permitted to rotate, and when all the joints to the left of B are free to deflect and rotate subject to the elastic restraints of the members and supports; in this case, S'_{Byz} is the moment per unit of deflection necessary to prevent rotation at joint B when it is deflected.

The sign convention indicated in Figure 12 was used throughout the following derivations. A moment on the right face of a free body is considered positive when clockwise. A shear on the right face of a free body is considered positive when acting upward. An angle of rotation is considered positive when clockwise. A deflection is considered positive when the joint is deflected upward.

The quantities S_{Bzz}, S_{Byz}, S'_{Byz}, and S_{Byy} were derived first for the case shown in Figure 13(a). The joint at A is hinged and fixed against translation, and the member AB is the first member in a series as is shown in Figure 6(a).

Referring to Figure 13(b) and to equations (28) and (29) the following equations for the moments at both ends of the member can be written:

$$M_{AB} = S_1(-\theta_A + C_1\alpha) = 0 \qquad (85)$$

$$M_{BA} = S_1(\alpha - C_1\theta_A) \qquad (86)$$

Solving equation (85) for θ_A results in

$$\theta_A = C_1\alpha \qquad (87)$$

Substituting equation (87) in equation (86) results in

$$M_{BA} = S_1(1 - C_1^2)\alpha \qquad (88)$$

By definition,

$$S_{Bzz} = \frac{M_{BA}}{\alpha} = S_1(1 - C_1^2) \qquad (89)$$

Referring to Figure 13(b) and summing moments about A:

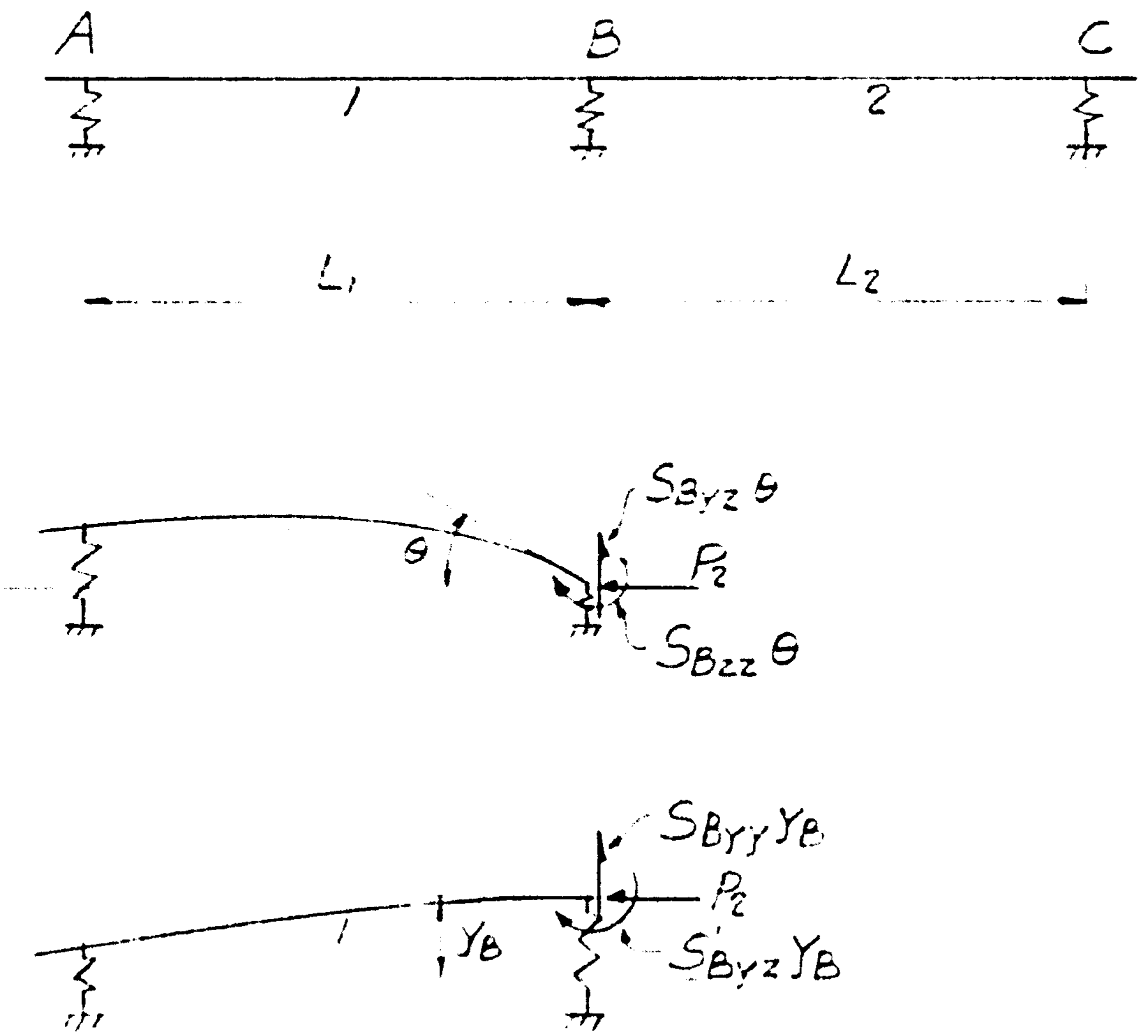


Figure 12. Sign convention for the stiffness criterion. Positive quantities shown.

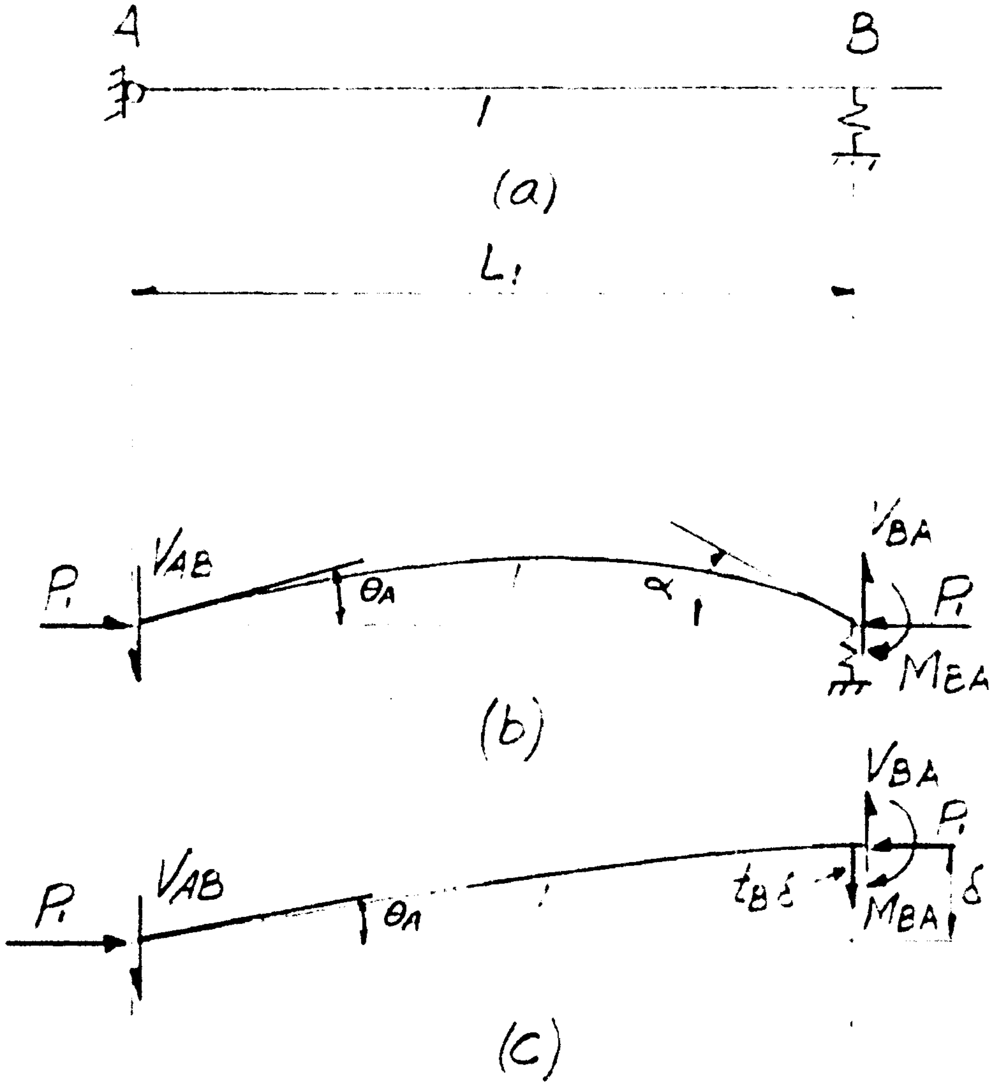


Figure 13. First span of a beam-column member with End-Joint A hinged.

$$M_{BA} - V_{BA}L_1 = 0 \qquad (90)$$

Substituting equation (88) in equation (90) results in

$$V_{BA} = \frac{S_1}{L_1}(1 - C_1^2)\alpha \qquad (91)$$

By definition,

$$S_{Byz} = \frac{V_{BA}}{\alpha} = \frac{S_1}{L_1}(1 - C_1^2) = \frac{S_1'''}{L_1}(1 - C_1) \qquad (92)$$

Referring to Figure 13(c) and to equations (28) and (29) the following equations for the moments at both ends of the member can be written:

$$M_{AB} = S_1\left[- \theta_A + \frac{(1 + C_1)\delta}{L_1}\right] = 0 \qquad (93)$$

$$M_{BA} = S_1\left[-C_1\theta_A + \frac{(1 + C_1)\delta}{L_1}\right] \qquad (94)$$

Solving equation (93) for θ_A results in

$$\theta_A = \frac{(1 + C_1)\delta}{L_1} \qquad (95)$$

Substituting equation (95) in equation (94) results in

$$M_{BA} = \frac{S_1'''}{L_1}(1 - C_1)\delta \qquad (96)$$

By definition,

$$S_{Byz}' = \frac{M_{BA}}{\delta} = \frac{S_1'''}{L_1}(1 - C_1) \qquad (97)$$

Comparing equations (92) and (97), S_{Byz}' is found to equal S_{Byz} and therefore either quantity will hereafter be referred to as S_{Byz}.

Referring to Figure 13(c) and summing moments about A:

$$M_{BA} + t_B \delta L_1 - V_{BA} L_1 - P_1 \delta = 0 \tag{98}$$

Substituting equations (94) and (95) in equation (98) results in

$$V_{BA} = t_B \delta + T_1 \delta - \frac{S_1'''}{L_1^2} (1 + C_1) \delta \tag{99}$$

By definition,

$$S_{Byy} = \frac{V_{BA}}{\delta} = t_B + T_1 - \frac{S_1'''}{L_1^2} (1 + C_1) \tag{100}$$

Next, a more general case was taken as is shown in Figure 14. The quantities S_{Czz}, S_{Cyz}, and S_{Cyy} were derived for joint C when the left end of the member, joint B in this case, is elastically restrained against rotation and deflection instead of hinged as in Figure 13.

Referring to Figure 14(b) and to equations (28) and (29) the following equations for the moments at both ends of the member can be written:

$$M_{BC} = S_2 \left[\theta + C_2 \alpha - \frac{(1 + C_2) y_B}{L_2} \right] \tag{101}$$

$$M_{CB} = S_2 \left[\alpha + C_2 \theta - \frac{(1 + C_2) y_B}{L_2} \right] \tag{102}$$

Referring to Figure 14(c) and combining the quantities by the principle of superposition, the moment and shear at B can be expressed in terms of S_{Bzz}, S_{Byz}, and S_{Byy}.

$$M_{BA} = S_{Bzz} \theta + S_{Byz} y_B \tag{103}$$

$$V_{BA} = S_{Byy} y_B + S_{Byz} \theta \tag{104}$$

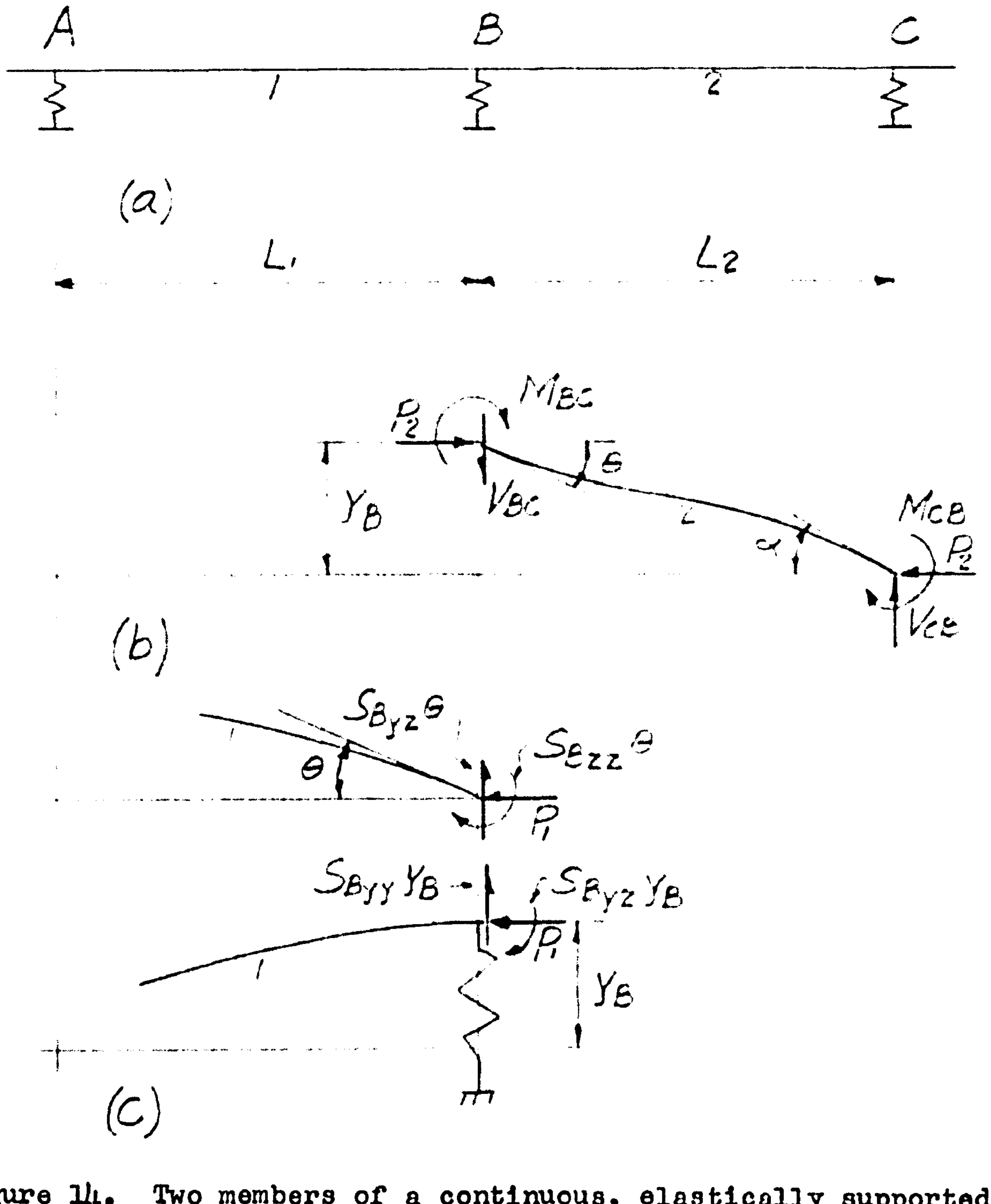


Figure 14. Two members of a continuous, elastically supported beam-column with Joints A and B restrained against rotation and deflection, and with Joint C restrained against rotation and fixed against deflection.

From the assumed sign convention $M_{BC} = - M_{BA}$. Therefore,

$$- S_{Bzz}\theta - S_{Byz}y_B = S_2\theta + S_2C_2\alpha - \frac{S_2'''}{L_2} y_B \qquad (105)$$

Referring to Figure 14(b) and summing moments about C:

$$M_{BC} + M_{CB} + P_1y_B - V_{BC}L_2 = 0 \qquad (106)$$

Substituting equations (101) and (102) and the relationship $V_{BC} = V_{BA}$ in equation (106) results in

$$S_2'''\alpha + S_2'''\theta - \left[\frac{2S_2'''}{L_2} - P_2\right] y_B - \left[S_{Byy}y_B + S_{Byz}\theta\right] L_2 = 0 \qquad (107)$$

Making the following substitutions:

$$A_2 = S_2 + S_{Bzz} \qquad (108)$$

$$B_2 = \frac{S_2'''}{L_2} - S_{Byz} \qquad (109)$$

$$D_2 = T_2 + S_{Byy} \qquad (110)$$

and using equation (11), equations (105) and (106) become

$$S_2C_2\alpha + A_2\theta - B_2y_B = 0 \qquad (111)$$

$$\frac{S_2'''}{L_2}\alpha + B_2\theta - D_2y_B = 0 \qquad (112)$$

Equations (111) and (112) are solved simultaneously for θ and y_B, and these values are substituted in equation (102) to solve for M_{CB}. The moment, M_{CB}, by definition, is $S_{Czz}\alpha$. The quantities θ and y_B vary linearly with respect to α and therefore M_{CB} varies linearly with α.

$$S_{Czz} = \frac{M_{CB}}{\alpha} = S_2 + \frac{(S_2C_2)^2D_2 - 2\,S_2C_2\frac{S_2'''}{L_2}\,B_2 + \left(\frac{S_2'''}{L_2}\right)^2 A_2}{B_2^2 - A_2D_2} \qquad (113)$$

By definition, $V_{CB} = S_{Cyz}\,\alpha$. Substituting for θ and y_B in equation (104) results in

$$S_{Cyz} = \frac{V_{CB}}{\alpha} = \frac{V_{BA}}{\alpha} = \frac{S_{Byz}S_2C_2T_2 - S_{Byy}\frac{S_2'''}{L_2}\,(A_2 - S_2C_2) - S_{Byz}\,\frac{S_2'''}{L_2}\,B_2}{B_2^2 - A_2D_2} \qquad (114)$$

Referring to Figure 15(b) and to equations (28) and (29) the following equations for the moments at both ends of the member can be written:

$$M_{BC} = S_2\left[-\theta + (1 + C_2)\frac{\delta - y_B}{L_2}\right] \qquad (115)$$

$$M_{CB} = S_2\left[-C_2\theta + (1 + C_2)\frac{\delta - y_B}{L_2}\right] \qquad (116)$$

Referring to Figure 15(c) and combining the quantities by the principle of superposition, the moment and shear at B can be expressed in terms of S_{Bzz}, S_{Byz}, and S_{Byy}.

$$M_{BA} = S_{Byz}y_B - S_{Bzz}\theta \qquad (117)$$

$$V_{BA} = S_{Byy}y_B - S_{Byz}\theta \qquad (118)$$

From the assumed sign convention $M_{BC} = -M_{BA}$. Therefore

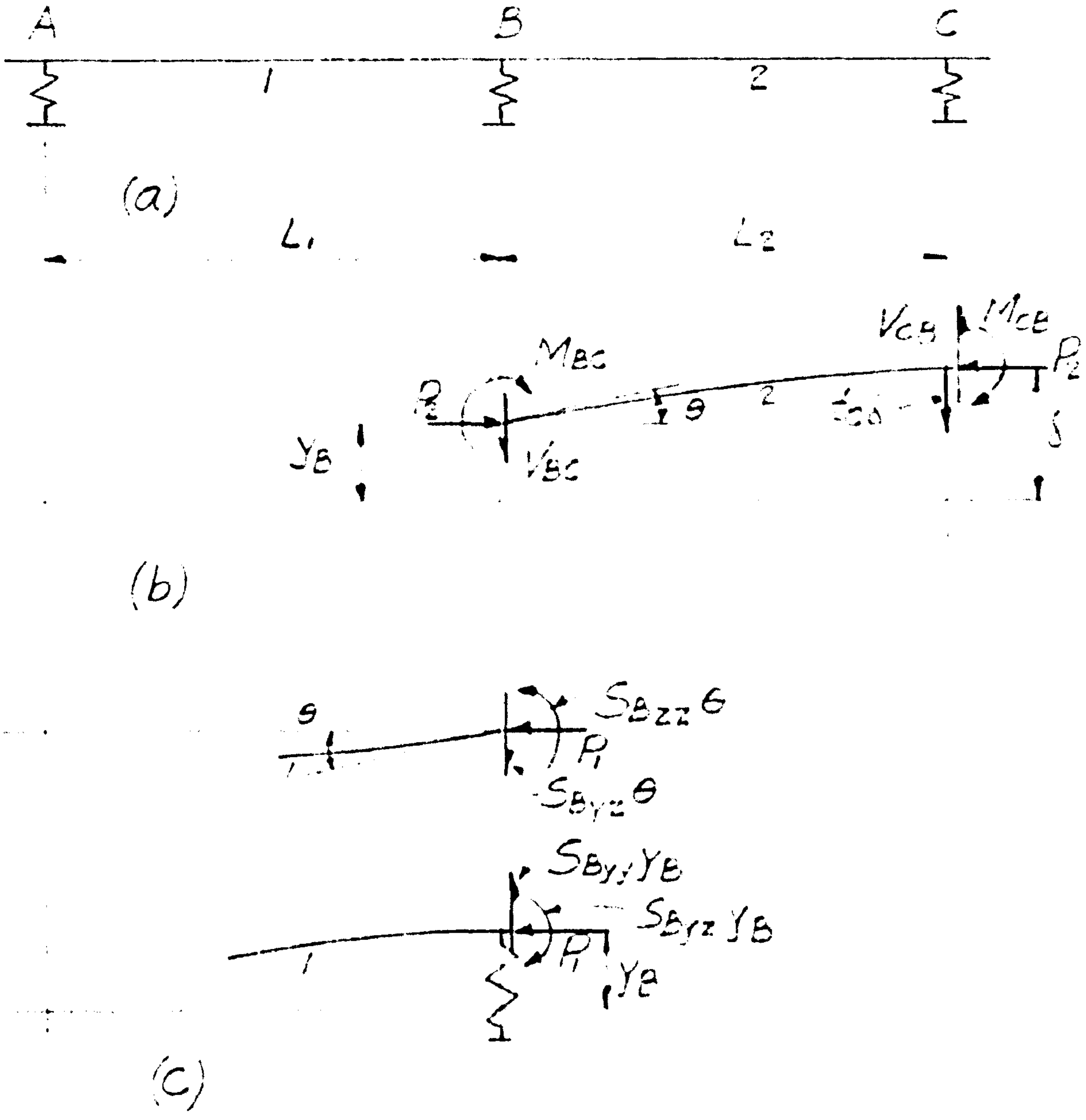


Figure 15. Two members of a continuous, elastically supported beam-column with Joints A and B restrained against rotation and deflection, and with Joint C restrained against deflection and fixed against rotation.

$$- S_{Byz}y_B + S_{Bzz}\theta \ = - S_2 + \frac{S_2'''}{L_2}(\delta \ - y_B) \quad (119)$$

Referring to Figure 15(b) and summing moments about C:

$$M_{BC} + M_{CB} - P_2(\delta \ - y_B) - V_{BC}L_2 = 0 \quad (120)$$

Substituting equation (115) and (116) and the relationship $V_{BA} = V_{BC}$ in equation (119) results in

$$- S_2'''\theta \ + \left[\frac{2S_2'''}{L_2} - P_2\right] (\delta \ - y_B) - (S_{Byy}y_B - S_{Byz}\theta)L_2 \ = 0 \quad (121)$$

Making substitutions from equations (108), (109), (110), and (11), equations (119) and 121) become

$$A_2\theta \ + B_2y_B - \frac{S_2'''}{L_2}\delta \ = 0 \quad (122)$$

$$T_2\delta \ - B_2\theta \ - D_2y_B = 0 \quad (123)$$

Equations (122) and (123) are solved simultaneously for θ and y_B, and these values substituted in equation (118) to solve for V_{BA}. The value of V_{BA} varies linearly with $\propto$.

Referring to Figure 15 and summing forces in the y direction:

$$V_{BC} + t_C\delta \ - V_{CB} = 0$$

$V_{BC} = V_{BA}$ and by definition V_{CB} is $S_{Cyy}\ \delta$. Therefore,

$$S_{Cyy} = \frac{V_{CB}}{\delta} = t_C + \frac{\left(\frac{S_2'''}{L_2}\right)^2 S_{Byy} + T_2(S_{Byz})^2 - T_2A_2S_{Byy}}{B_2^2 - A_2D_2} \quad (124)$$

By definition, $M_{CB} = S_{Cyz}\,\delta$. Substituting for θ and y_B in equation (116) results in

$$S_{Cyz} = \frac{M_{CB}}{\delta} =$$

$$\frac{S_{Byz}S_2C_2T_2 - S_{Byy}\dfrac{S_2'''}{L_2}(A_2 - S_2C_2) - \dfrac{S_2'''}{L_2}S_{Byz}B_2}{B_2^2 - A_2D_2} \tag{125}$$

By comparing equations (92) with (97) and (114) with (125) a reciprocal relationship is found to exist for the conditions used in the derivations of these equations. This reciprocal relation can be stated as follows:

When a moment is applied at the right end of a member that is elastically restrained against rotation and deflection at the left end, the force per unit of rotation required to prevent deflection is numerically equal to the moment per unit of deflection required to prevent rotation when the right end of the same member is deflected by a force.

The equations derived in this section for use in applying the stiffness criterion are used as follows:

1. Determine the rotational stiffness of the end of a member by using equation (89) or (113).

2. Determine the translational stiffness of the end of a member by using equation (100) or (124).
3. The quantity from equation (92) or (125) is used along with the stiffnesses obtained from equations (89) and (100) or (113) and (124) in subsequent computations to obtain the stiffnesses of the far end of the next member.

The equations for rotational and translational stiffnesses, as derived, can be used to determine joint stiffnesses under the following conditions:

1. The equations for rotational stiffness can be used at the end of the beam-column when the end is hinged and fixed against deflection and at a joint that is located at a node on the buckling curve. The rotational stiffness at the hinged end of a beam-column is determined directly with the equation for the rotational stiffness for the end of a member. The rotational stiffness at a joint that is located at a node on the buckling curve is obtained by adding algebraically the rotational stiffness obtained for the ends of the members that meet at the joint.
2. The equations for translational stiffness can be used at a joint that is located at a point of maximum deflection on the buckling curve. The

translational stiffness of the joint is determined by adding algebraically the translational stiffnesses obtained for the ends of the members that meet at the joint, being careful to add only once the translational stiffness of the elastic support located at the joint.

A numerical example is included in Appendix B to demonstrate the use of the stiffness criterion.

DESIGN AND STRESS DATA FOR THE THREE-SPAN CONTINUOUS PONY TRUSS BRIDGE

The three-span continuous pony truss bridge that was used as an example for the application of stability analysis was designed by the Iowa Highway Commission. The pertinent design data necessary for the stability analysis as previously outlined are shown in Figure 16.

The axial stresses were determined in this continuous truss on the basis of the usually accepted assumptions used in bridge design. These assumptions are:

1. The bridge is made up of a number of planar structures and each part is analyzed independently.
2. The superimposed loads are transmitted from the roadway to the trusses by simple beam action of the floor beams.
3. The end reactions of the floor beams are regarded as applied loads on the vertical trusses which are analyzed as continuous trusses.
4. The joints are pin-connected.
5. The dead load is applied as equal panel loads at the same points of application as the live loads.

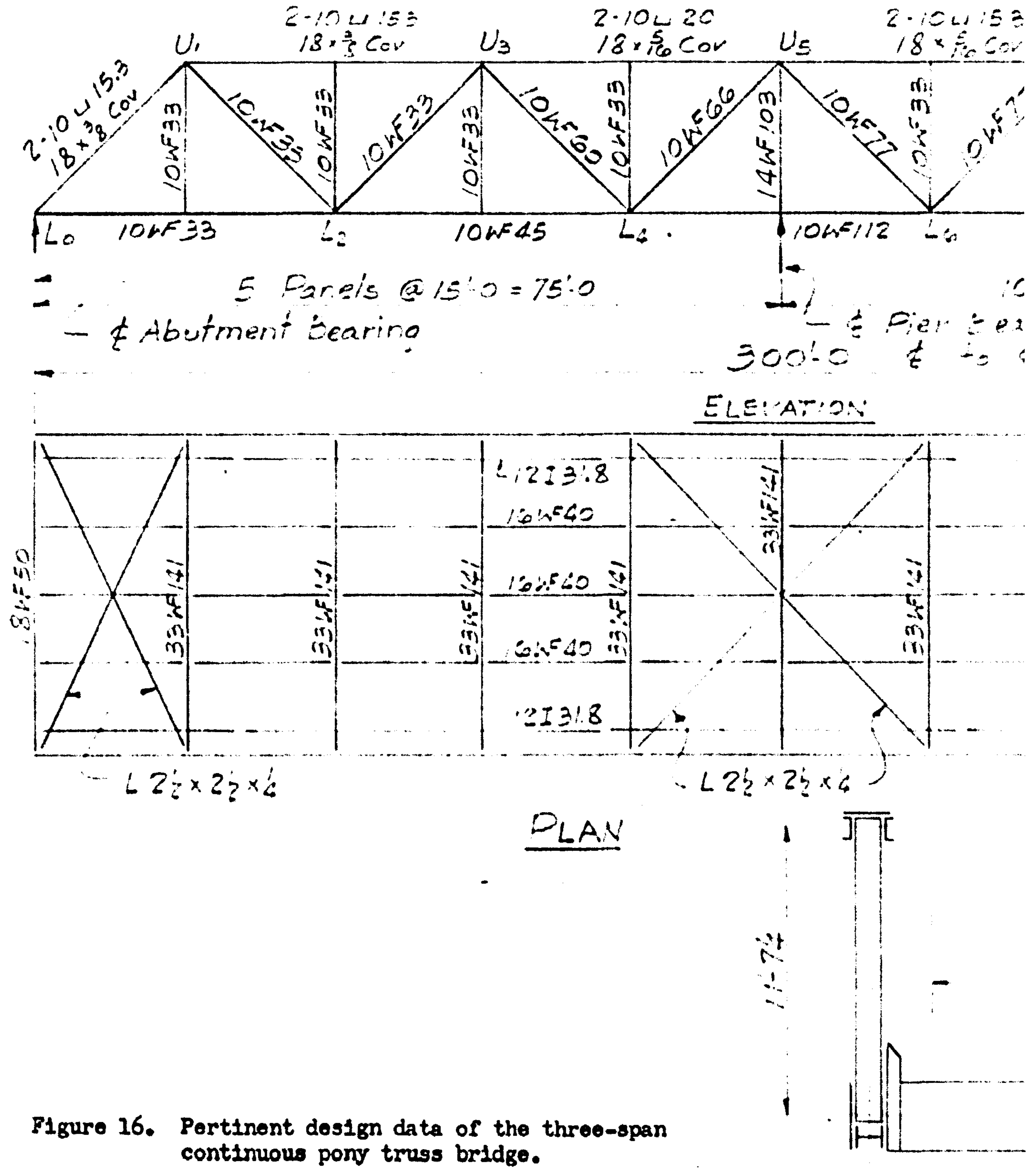
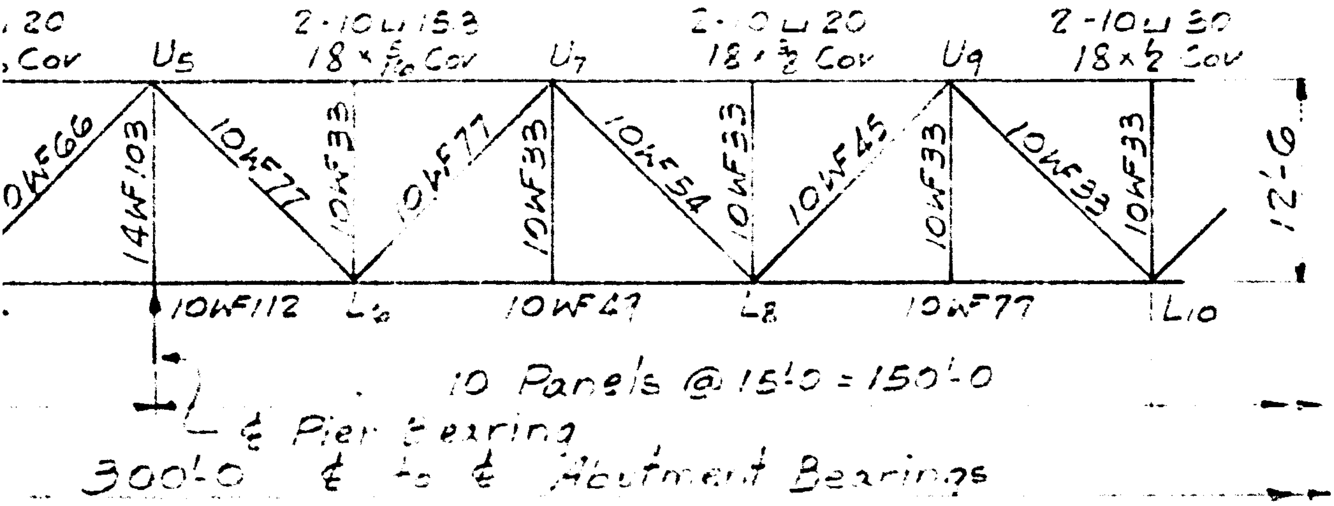
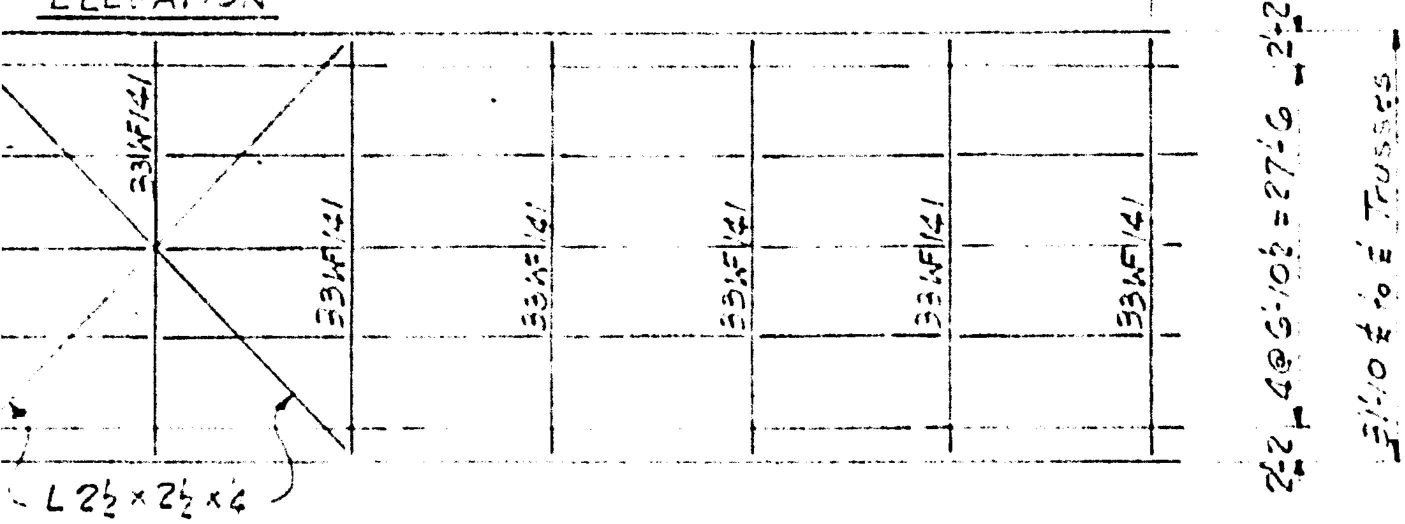
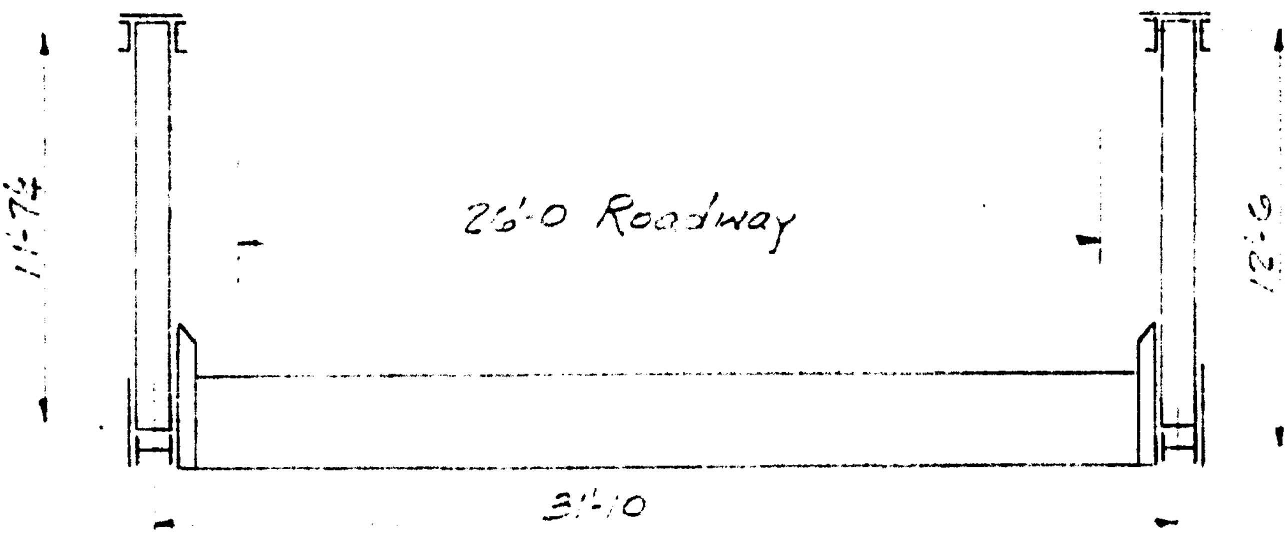


Figure 16. Pertinent design data of the three-span continuous pony truss bridge.



ELEVATION





CROSS SECTION

Based on the above assumptions, the structure analyzed was simplified to a continuous planar truss over four supports. The structure, therefore, is externally statically indeterminate to the second degree. The reactions for a unit load placed at the lower chord panel points were determined using the strain-energy theory. With the reactions determined, an influence table was made for the stresses in the end post and top chord members; these members make up the beam-column to be analyzed for stability. The influence table for the stresses is given in Table 1; a plus sign indicates a tensile stress and a minus sign indicates a compressive stress. Since the structure is symmetrical, the stresses are tabulated for the members on one side of the center-line of the span.

The dead load stresses in the members were computed using a dead panel load determined from the design drawing of the Iowa Highway Commission. An estimate of the total dead weight was given and this quantity was assumed to be divided equally among all of the lower chord panel points of the bridge. On this basis, the dead panel load is 31.23 kips.

The dead load stresses in the top chord members are tabulated in Table 2.

The live load stresses in the truss members were determined using the loading requirements of the Standard Highway

Table 1. Influence Table for the Stresses in the End Post and Top Chord Members

Load At.	Members					
	L_0U_1	U_1U_3	U_3U_5	U_5U_7	U_7U_9	U_9U_{10}
L_1	-1.1872	-0.6240	-0.0480	+0.2098	+0.1492	+0.0886
L_2	-0.8257	-1.2686	-0.1373	+0.3746	+0.2664	+0.1583
L_3	-0.4927	-0.7570	-0.3139	+0.4436	+0.3156	+0.1876
L_4	-0.2216	-0.3406	-0.6811	+0.3048	+0.2171	+0.1294
L_6	+0.1612	+0.2477	+0.4954	-0.5048	-0.3530	-0.2012
L_7	+0.2610	+0.4010	+0.8021	-0.0200	-0.8652	-0.5104
L_8	+0.2957	+0.4543	+0.9086	+0.2441	-1.5395	-0.9230
L_9	+0.2930	+0.4502	+0.9005	+0.3756	-1.1243	-1.4242
L_{10}	+0.2618	+0.4022	+0.8045	+0.4056	-0.7944	-1.9944
L_{11}	+0.2151	+0.3305	+0.6610	+0.3761	-0.5240	-1.4242
L_{12}	+0.1609	+0.2472	+0.4944	+0.3098	-0.3066	-0.9230
L_{13}	+0.0981	+0.1507	+0.3014	+0.1993	-0.1555	-0.5104
L_{14}	+0.0464	+0.0713	+0.1426	+0.1024	-0.0494	-0.2012
L_{16}	-0.0234	-0.0360	-0.0720	-0.0461	+0.0416	+0.1294
L_{17}	-0.0345	-0.0530	-0.1061	-0.0685	+0.0595	+0.1876
L_{18}	-0.0292	-0.0449	-0.0898	-0.0581	+0.0512	+0.1583
L_{19}	-0.0164	-0.0252	-0.0504	-0.0326	+0.0280	+0.0886
	-2.8307	-3.1493	-1.4986	-0.7301	-5.7119	-8.1120
	+1.7932	+2.7551	+5.5105	+3.3457	+1.1286	+1.1278
Total	-1.0375	-0.3942	+4.0119	+2.6156	-4.5833	-6.9842

Table 2. Dead Load Stresses

	Stress - kips					
Member	L_0U_1	U_1U_3	U_3U_5	U_5U_7	U_7U_9	U_9U_{10}
	-32.40	-12.31	+125.29	+81.68	-143.14	-218.12

Specifications (1). These Specifications require an equivalent loading consisting of a uniformly distributed load plus a concentrated load placed on the bridge so as to produce a maximum stress. The magnitude of the panel load computed from the uniformly distributed load is 11.41 kips; this panel load may be placed at as many panels as necessary to produce a maximum stress. The magnitude of the additional live panel load computed for the concentrated load is 21.39 kips; this panel load may be placed at one or two panel points depending on the location of the member in which the stress is being computed. In computing the panel loads due to live load the lane loading was shifted out of its half of the roadway width; the bridge was designed under this condition. The stresses due to impact are taken as a per cent, depending on the loaded length, of the live load stresses.

In making a stability analysis, the stresses in all the members must be determined for one position of the live load; therefore, only a few of the members will be stressed to their maximum live load stress.

The two conditions of loading that will be considered for the stability analysis are: The loading that will produce a maximum compressive stress in the member U_1U_3; and the loading that will produce the maximum compressive stress in the member U_9U_{10}. Hrennikoff (17) has shown that these conditions would be critical as far as buckling of the top chord is concerned.

The position of the live load that produces a maximum compressive stress in U_1U_3 is obtained by placing the live panel load from the uniformly distributed load at all of the panel points of the two end spans and by placing the two concentrated loads at L_2 and L_{18} to maintain a condition of symmetry. The live load and impact stresses in the end post and top chord members for this placing of the live load are tabulated in Table 3.

Table 3. Live Load Stresses for the Position of Live Load that Produces a Maximum Stress in the Member U_1U_3

	Stress - kips					
Member	L_0U_1	U_1U_3	U_3U_5	U_5U_7	U_7U_9	U_9U_{10}
Unif. L. L.	-32.30	-35.93	-17.10	+12.86	+12.88	+12.87
Conc. L. L.	-18.29	-28.10	- 4.86	+6.77	+6.79	+6.77
Total	-50.59	-64.03	-21.96	+19.63	+19.67	+19.64
Impact	-12.65	-16.01	- 5.49	+4.91	+4.92	+4.91
Total L. L.	-63.24	-80.04	-27.45	+24.54	+24.59	+24.55

The position of the live load that produces a maximum stress in the member U_9U_{10} is obtained by placing the live panel load from the uniformly distributed load at all of the panel points in the center span and by placing the one concentrated load at L_{10}. The live load and impact stresses in the end post and top chord members for this placing of the live load are tabulated in Table 4.

Table 4. Live Load Stresses for the Position of Live Load that Produces a Maximum Stress in the Member U_9U_{10}

	Stress - kips					
Member	L_0U_1	U_1U_3	U_3U_5	U_5U_7	U_7U_9	U_9U_{10}
Unif. L. L.	+20.46	+31.44	+62.87	+16.98	-65.17	- 92.56
Conc. L. L.	+5.60	+8.60	+17.21	+8.68	-16.99	- 42.66
Total	+26.06	+40.04	+80.08	+25.66	-82.16	-135.22
Impact	+4.74	+7.28	+14.56	+4.66	-14.94	- 24.58
Total L. L.	+30.80	+47.32	+94.64	+30.32	-97.10	-159.80

The top chord members will be analyzed for stability for stress conditions computed on the basis of a load factor, applied to the live load stresses, that will produce a maximum unit stress in the members that is less than the yield strength of the material.

A load factor of 6 is used for the position of live load that produces a maximum stress in the member U_1U_3. The results are tabulated in Table 5.

A load factor of 3.5 is used for the position of live load that produces a maximum stress in the member U_9U_{10}. The results are tabulated in Table 6.

Table 5. Combined Dead Load Plus the Load Factor Times the Live Load for Maximum Stress in U_1U_3*

	Stress - kips					
Member	L_0U_1	U_1U_3	U_3U_5	U_5U_7	U_7U_9	U_9U_{10}
D. L.	- 32.40	- 12.31	+125.29	+81.68	-143.14	-218.12
(L. L.)(6)	-379.44	-480.24	-164.70	+147.24	+147.54	+147.30
Total	-411.84	-492.55	- 39.41	+228.92	+4.40	- 70.82

*Values obtained from Tables 2 and 3

Table 6. Combined Dead Load Plus the Load Factor Times the Live Load for Maximum Stress in U_9U_{10}*

	Stress - kips					
Member	L_0U_1	U_1U_3	U_3U_5	U_5U_7	U_7U_9	U_9U_{10}
D. L.	-32.40	-12.31	+125.29	+81.68	-143.14	-218.12
(L. L.)(3.5)	+107.80	+165.62	+331.24	+106.12	-339.85	-559.30
Total	+75.40	+153.31	+456.53	+187.80	-482.99	-777.42

*Values obtained from Tables 2 and 4.

The stresses given in Table 5 and 6 will be used in the stability analysis of the top chord member. For the stresses given in Table 5, the maximum unit stress occurs in member U_1U_3 with a value of 31,400 psi. For the stresses given in Table 6, the maximum unit stress occurs in member U_9U_{10} with a value of 29,200 psi. Since the yield strength of structural steel is approximately 35,000 psi., all of the unit stresses in the top chord members are within the yield strength of the material.

STABILITY ANALYSIS OF THE TOP CHORD OF THE THREE-SPAN CONTINUOUS PONY TRUSS BRIDGE

The series and stiffness criteria were applied in the stability analysis of the top chord of the three-span continuous pony truss bridge shown in Figure 16.

The top chord of a three-span continuous truss bridge has regions of tensile and compressive stress. The regions in which the members have a compressive stress can be analyzed for stability in the same manner as a structure in which the entire top chord is in compression provided that the series and stiffness criteria are applied at one of the joints in the region of compression. The regions of tensile stress adjoining the regions of compressive stress have a stiffening effect on the members in compression and have a tendency to reduce the possibility of instability.

The stresses in the top chord members were computed, in the preceding section, for two live loading conditions. The top chord was checked for stability for these two stress situations in the following manner:

1. For the loading condition that produces a maximum compressive stress in the region U_1U_3, the series criterion was used by applying an external moment at the joint L_0 and the stiffness criterion was used

by determining the rotational stiffness of the joint L_0.

2. For the loading condition that produces a maximum compressive stress in the region U_9U_{10}, the stiffness criterion was used by determining the rotational and translational stiffnesses of the joint U_{10}.

Elastic Properties of the End Post

The procedures of stability analysis as presented in this thesis have been developed for a member that forms a straight line. The elastic properties of the sloping end post member L_0U_1 of the pony truss must, therefore, be expressed in terms of its horizontal projection.

The end post member, L_0U_1, is shown in Figure 17(a). The member has an actual length of L_a and a horizontal projection of length L. The quantities S, C and T are determined for the horizontal projection of the member as shown in Figure 17(b).

In the following equations a quantity like M_z represents a moment about the z-axis and α_z represents an angle about the z-axis, both the moment and the angle are in the x-y plane.

Applying a moment, Figure 17(b), of M_z at U_1 producing

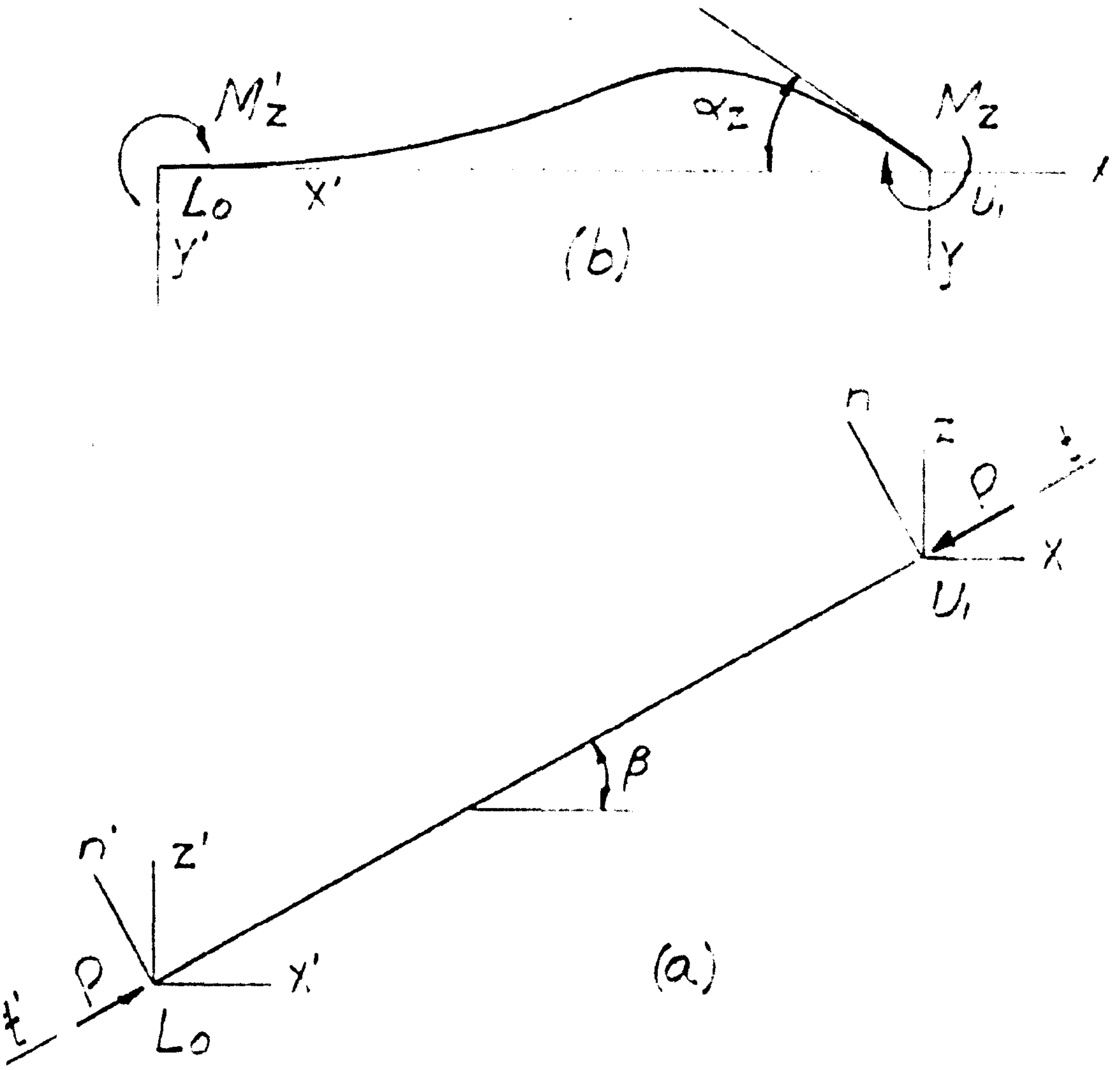


Figure 17. Transfer of elastic constants of the end post member to horizontal equivalents.

an angle of rotation of α_z results in

$$\alpha_n = \alpha_z \mathrm{Cos}\beta \tag{126}$$

$$\alpha_t = \alpha_z \mathrm{Sin}\beta \tag{127}$$

The moment M_z can be expressed as

$$M_z = M_n \mathrm{Cos}\beta + M_t \mathrm{Sin}\beta \tag{128}$$

in which

$$M_n = S_r \alpha_n \tag{129}$$

$$M_t = S_t \alpha_t \tag{130}$$

The quantity S_r is the stiffness factor determined from equation (5) using the inclined length of the member. The quantity S_t is the torsional stiffness of the member. A solution for the torsional stiffness of the end post member is included in Appendix C.

Substituting equations (126) and (127) in equations (129) and (130), and substituting these results in equation (128) results in

$$M_z = S_r \alpha_z \mathrm{Cos}^2\beta + S_t \alpha_z \mathrm{Sin}^2\beta \tag{131}$$

The stiffness factor, S, is the moment per unit of rotation; therefore,

$$S = \frac{M_z}{\alpha_z} = S_r \mathrm{Cos}^2\beta + S_t \mathrm{Sin}^2\beta \tag{132}$$

The moments induced at L_0, Figure 17(a), can be expressed as

$$M_n' = C_1 M_n \tag{133}$$

$$M_t' = - M_t \tag{134}$$

The quantity C_1 is the carry-over factor determined from equation (4) using the inclined length of the member.

The moment M_z' can be expressed as

$$M_z' = M_n' \cos\beta - M_t' \sin\beta \tag{135}$$

or

$$M_z' = C_1 S_r \alpha_z \cos^2\beta - S_t \alpha_z \sin^2\beta \tag{136}$$

The carry-over factor, C, is therefore

$$C = \frac{M_z'}{M_z} = \frac{C_1 S_r \cos^2\beta - S_t \sin^2\beta}{S_r \cos^2\beta + S_t \sin^2\beta} \tag{137}$$

The translational stiffness, T, of the end post computed on the basis of the inclined length is also the translational stiffness, T, for the horizontal projection of the member.

Stiffness of the Elastic Supports

The top chord members are elastically supported in a lateral direction at the panel points by a frame composed of the verticals of the truss and the floor beams framing into the verticals.

A line diagram representing the frame at each panel point is shown in Figure 18(a). The force, P, per unit of deflection, δ , is defined as the stiffness, t, of the elastic support.

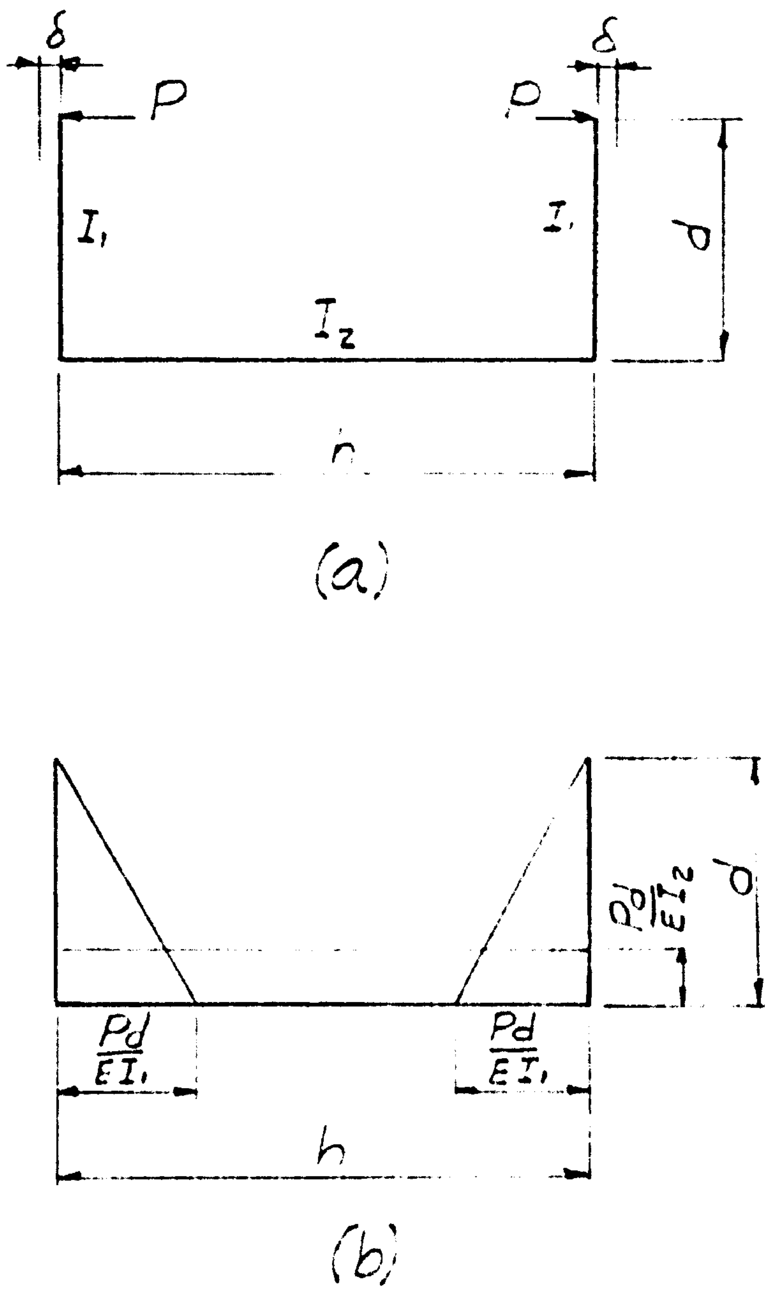


Figure 18. Diagrams representing the cross frame at each panel point.

The moment-area method is used to determine the value of the stiffness of the elastic support. Using the $\frac{M}{EI}$ diagram shown in Figure 18(b), the equation for the deflection is

$$\delta = \frac{Pd}{EI_1}\,\frac{d}{2}\,\frac{2d}{3} + \frac{Pd}{EI_2}\,\frac{h}{2}\,d \qquad (138)$$

Therefore,

$$t = \frac{P}{\delta} = \frac{1}{\dfrac{d^3}{3EI_1} + \dfrac{hd^2}{2EI_2}} \qquad (139)$$

The stiffness of the elastic support at U_5 and U_{15} is determined with equation (139) and information from Figure 16. The quantities required for equation (139) are

$I_1 = 1165.8$ in.^{4}

$I_2 = 7442.2$ in.^{4}

$E = 29{,}000{,}000$ psi.

$d = 11.604$ ft. (11'-7¼)

$h = 31.833$ ft. (31'-10)

resulting in $t = 22.84$ kips per inch.

The stiffness of the elastic supports at all of the other intermediate panel points, at which $I_1 = 170.9$ in.^{4}, is 5.03 kips per inch.

Numerical Results of Stability Analysis

First, the top chord members were checked for stability using the stresses given in Table 5. These stresses were determined from the loading condition, with a load factor of 6, that produced a region of compressive stress near the end of the span with a maximum value in the member U_1U_3.

If the region in compression buckles, the member at L_0 must rotate. Therefore, the series and stiffness criteria were applied to joint L_0.

The quantities that were computed for use in applying the series criterion are tabulated in Table 7. A moment of 100 inch kips was applied at joint L_0. All of the joints except L_0 were balanced and the summation of the moments carried back to L_0 was found to be 29.7 inch kips resulting in a value of the factor r of 0.297.

The quantities C, S, and T in Table 7 for the members were used in applying the stiffness criterion. The solution was made by starting at one end of the structure and working to the far end. The rotational stiffness of joint L_0, in considering the entire structure to the right, was found to be 66,000 inch kips per radian of rotation.

Second, the top chord members were checked for stability using the stresses given in Table 6. These stresses were determined from the loading condition, with a load factor of

Table 7. Computed Quantities for Use in Applying the Series and Stiffness Criteria

Member	I	Stress	C	S	T	S_L	S_R	C_{aL}	C_a
	in.^{4}	kips		in. kips / radian	kips / in.	in. kips / radian	in. kips / radian		
L_0U_1	537.80	-411.84	0.508	152,480	12.43	94,080	141,680	0.203	0
U_1U_2	537.80	-492.55	0.527	334,590	28.80	165,990	162,920	0.046	0
U_2U_3	537.80	-492.55	0.527	334,590	28.80	164,450	163,940	0.039	0.
U_3U_4	640.73	-39.41	0.502	412,060	37.98	205,340	208,440	0.0	0.
U_4U_5	640.73	-39.41	0.502	412,060	37.98	217,100	191,110	0.055	-0.
U_5U_6	528.71	+228.92	0.488	346,200	33.07	182,380	204,020	0.028	0.
U_6U_7	528.71	+228.92	0.488	346,200	33.07	197,840	196,290	0.105	0.
U_7U_8	671.33	+4.40	0.500	432,770	40.09	209,460	210,760	-0.033	-0.
U_8U_9	671.33	+4.40	0.500	432,770	40.09	211,410	212,710	-0.025	-0.
U_9U_{10}	986.04	-70.82	0.502	633,720	58.37	262,500	263,450	-0.203	-0.

Applying the Series and Stiffness Criteria

...ips/...n	T kips/in.	S_L in. kips/radian	S_R in. kips/radian	C_{aL}	C_{aR}	C_{bL}	C_{bR}	C_{cL}	C_{cR}
...30	12.43	94,080	141,680	0.203	0.470	0.503			
...90	28.80	165,990	162,920	0.046	0.029	0.466	0.492	0.696	0.742
...90	28.80	164,450	163,940	0.039	0.035	0.452	0.454	0.679	0.683
...60	37.98	205,340	208,440	0.0	0.016	0.397	0.378	0.780	0.731
...60	37.98	217,100	191,110	0.055	-0.073	0.166	0.324	0.310	0.685
...00	33.07	182,380	204,020	0.028	0.131	0.319	0.178	0.694	0.344
...00	33.07	197,840	196,290	0.105	0.097	0.371	0.383	0.740	0.769
...70	40.09	209,460	210,760	-0.033	-0.027	0.347	0.338	0.718	0.695
...70	40.09	211,410	212,710	-0.025	-0.016	0.361	0.350	0.740	0.713
...20	58.37	262,500	263,450	-0.203	-0.197	0.298	0.290	0.726	0.702

Table 8. Computed Quantities for Use in Applying the Stiffness Criterion.

Member	I in.^{4}	Stress kips	C	S $\frac{\text{in. kips}}{\text{radian}}$	T $\frac{\text{kips}}{\text{in.}}$
L_0U_1	537.80	+75.40	0.464	161,610	14.93
U_1U_2	537.80	+153.31	0.492	350,270	33.11
U_2U_3	537.80	+153.31	0.492	350,270	33.11
U_3U_4	640.93	+456.53	0.481	423,900	41.29
U_4U_5	640.93	+456.53	0.481	423,900	41.29
U_5U_6	528.71	+187.80	0.490	345,230	32.79
U_6U_7	528.71	+187.80	0.490	345,230	32.79
U_7U_8	671.33	-482.79	0.521	420,900	36.84
U_8U_9	671.33	-482.79	0.521	420,900	36.84
U_9U_{10}	986.04	-777.42	0.523	616,500	53.64

3.5, that produced a region of compressive stress in the center portion of the span with a maximum value in the member U_9U_{10}.

If the region in compression buckles, the joint at L_{10} would be either a point of maximum deflection or a point of a node on the deflection curve due to the symmetry of the structure and the stress condition.

The quantities that were computed for use in applying the stiffness criterion are tabulated in Table 8. The solution was made by starting at one end of the structure and working to the joint L_{10}. In considering the entire structure to the right and left of joint L_{10}, the rotational stiffness of joint L_{10} was found to be 331,400 inch kips per radian of rotation and the translational stiffness of joint L_{10} was found to be 11.31 kips per inch of deflection.

Discussion of Numerical Results

In using the series criterion in checking for stability for the first condition of loading, the stability factor, r, was found to be 0.297. When the stability factor is less than one the moment distribution series converges and a condition of stability is indicated. In making the moment distribution solution the moments that were carried over to the joints beyond U_6 were of such magnitude that when these joints were balanced, no moment was carried back to joint L_0 and therefore did not contribute anything to the series solution. Therefore, for this particular solution, only a part of the structure needed to be considered in applying the series criterion at joint L_0.

In using the stiffness criterion in checking for stability for the first condition of loading the rotational stiffness of joint L_0 was found to be 66,000 inch kips per radian of rotation; the moment was found to be in the same direction as the rotation indicating a condition of stability which agrees with the series solution.

The series criterion solution can be used to obtain the same numerical result as was obtained from the stiffness criterion solution. For the left end of member L_0U_1, the stiffness factor, S_L, as given in Table 7 is 94,080 inch kips per radian of rotation; this is the moment necessary to rotate joint L_0 when the other joints are permitted to deflect subject to the elastic supports but are fixed against rotation. In the series solution it was found that for each 100 inch kips of moment applied at L_0, 29.7 inch kips was carried back from the other joints leaving 70.3 inch kips when all of the joints except L_0 had been balanced. Therefore, the moment necessary to rotate the member at L_0 one radian would be 0.703(94,080) = 66,140 inch kips which is essentially the moment determined from the stiffness criterion solution.

Since the same information can be obtained by the stiffness criterion as was obtained from the series criterion, only the stiffness criterion was used in checking for stability for the second condition of loading. The rotational stiffness of joint L_{10} was found to be 331,400 inch kips per radian of rotation. The moment was found to be in the same direction as the rotation indicating a condition of stability if the structure buckles with a node at L_{10} on the deflection curve. The translational stiffness of joint L_{10} was found to be 11.31 kips per inch of deflection. The force was found to be in the same direction as the deflection indicating a condition of stability if the structure buckles with a deflection curve which has a point of maximum deflection at L_{10}. Therefore, the top chord members were found to be stable for the possible buckling curves.

From the above results, it can be concluded that the top chord would not buckle with load factors which produce stresses in the members that are within the elastic strength of the material.

SUMMARY AND RECOMMENDATIONS

Summary

The object of this investigation was to make an analytical study of the elastic stability of the top chord of a three-span continuous pony truss bridge.

The top chord and end post members were considered hinged and fixed against lateral deflection at the ends of the span and elastically supported against lateral deflection at the intermediate panel points. The elastic supports were provided by the cross frames composed of the verticals of the truss and the floor beams framing into the verticals. The entire effect of the diagonal web members and the torsional stiffness of the vertical members were neglected. Also, it was assumed that the top chord members were not deflected out of a straight line, when viewed from above, by the loads applied to the floor system.

Two criteria, series and stiffness, were developed and were used as a basis for checking the elastic stability of the top chord members.

The series in the series criterion procedure is obtained from a moment distribution type of solution. Equations were derived for the necessary constants required for a moment distribution solution of a beam-column elastically supported against deflection at intermediate points. When the moment distribution series converges a condition of stability is indicated, but when the series diverges a condition of instability is indicated.

Equations were derived for use in applying the stiffness criterion to obtain the rotational and translational stiffnesses of one end of a member considering the entire structure beyond its far end. The equations, as derived, for the stiffnesses for the end of a member can be used to determine the rotational stiffness of the hinged end of a beam-column and the rotational and translational stiffnesses of a joint provided that the structure and stress conditions are exactly the same on both sides of the joint. The principles of the stiffness criterion are:

1. When an external clockwise moment, under a condition of equilibrium, is required at a joint to rotate the joint in a clockwise direction, there is an indication of stability; but, when the external moment necessary to hold the joint in equilibrium is opposite to the direction of rotation, then there is an indication of instability.
2. When a lateral force, under a condition of equilibrium, is applied at a joint and when the direction of the deflection of the joint is in the direction

of the applied force there is an indication of stability; but, when, under a condition of equilibrium, the direction of the force must be opposite to the direction of the deflection of the joint, then a condition of instability is indicated.

The process of determining the load factor that produces a stress condition which causes buckling must be one of trial; that is, the structure must be analyzed for stability with various load factors until a condition of instability is found or until the stresses reach the yield stress. In applying the series and stiffness criteria, the buckling curve corresponding to the minimum load factor must be obtained as was demonstrated in the illustrative example included in Appendix B. If the beam-column buckles with a node at one of the joints, the value of the translational stiffness of that joint may indicate a condition of stability; and, if the beam-column buckles with a point of maximum deflection at one of the joints, the rotational stiffness of that joint may indicate a condition of stability, and the stability factor determined with the use of the series criterion may indicate a condition of stability. Therefore, any results obtained by the use of the series and stiffness criteria must be carefully interpreted.

Of the two criteria, series and stiffness, the stiffness criterion involves a smaller number of equations and when it can be used it is the more practical of the two procedures. The series criterion can be applied to any joint and therefore can be considered as the more general procedure of the two. However, only the rotational stiffness of a joint can be checked by the series criterion procedure.

In checking the top chord of the three-span continuous pony truss bridge, the series and stiffness criteria must be applied to one of the joints in the region of compressive stress. The regions of tensile stress adjoining the regions of compressive stress have a stiffening effect on the members in compression and have a tendency to reduce the possibility of buckling.

A procedure was presented to determine the elastic constants for the inclined end post member in terms of the horizontal projection of the member. A similar procedure could be used to include the effect of the other diagonal web members in an analysis for stability. If the effect of the diagonal web members were included, the resistance of the top chord members to buckling would be increased.

The top chord of the three-span continuous pony truss was checked for stability for two conditions of live loading and the member was found to be stable with stresses in the top chord member which were within the elastic strength of the material.

Recommendations

On the basis of the work that has been done in this thesis the following extensions are suggested.

1. Determine the effect on the stability of the top chord member of the live load deflections of the floor beams and the resulting lateral movements of the tops of the vertical members.

2. Determine the forces which act at the top of the vertical members of the lateral cross frames as a result of the resistance offered to deflection by the top chord of the vertical trusses.

3. Extend the application of the series and stiffness criteria to the inelastic range of stress in the material. This might be used along with the principles of limit design to reduce the present factor of safety of design and thereby utilize the structural material more efficiently.

4. Consider the effects on the stability of the top chord member of the torsional resistance of the verticals and the entire resistance of the diagonals.

5. Determine what approximations can be made in applying the series and stiffness criteria to any given structure. Some indication of this was given for the particular structure used as an example in this investigation.

6. Extend the stiffness criterion so that the rotational and translational stiffnesses can be determined for an intermediate joint that is not at a point of symmetry in the structure.

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APPENDICES

Appendix A

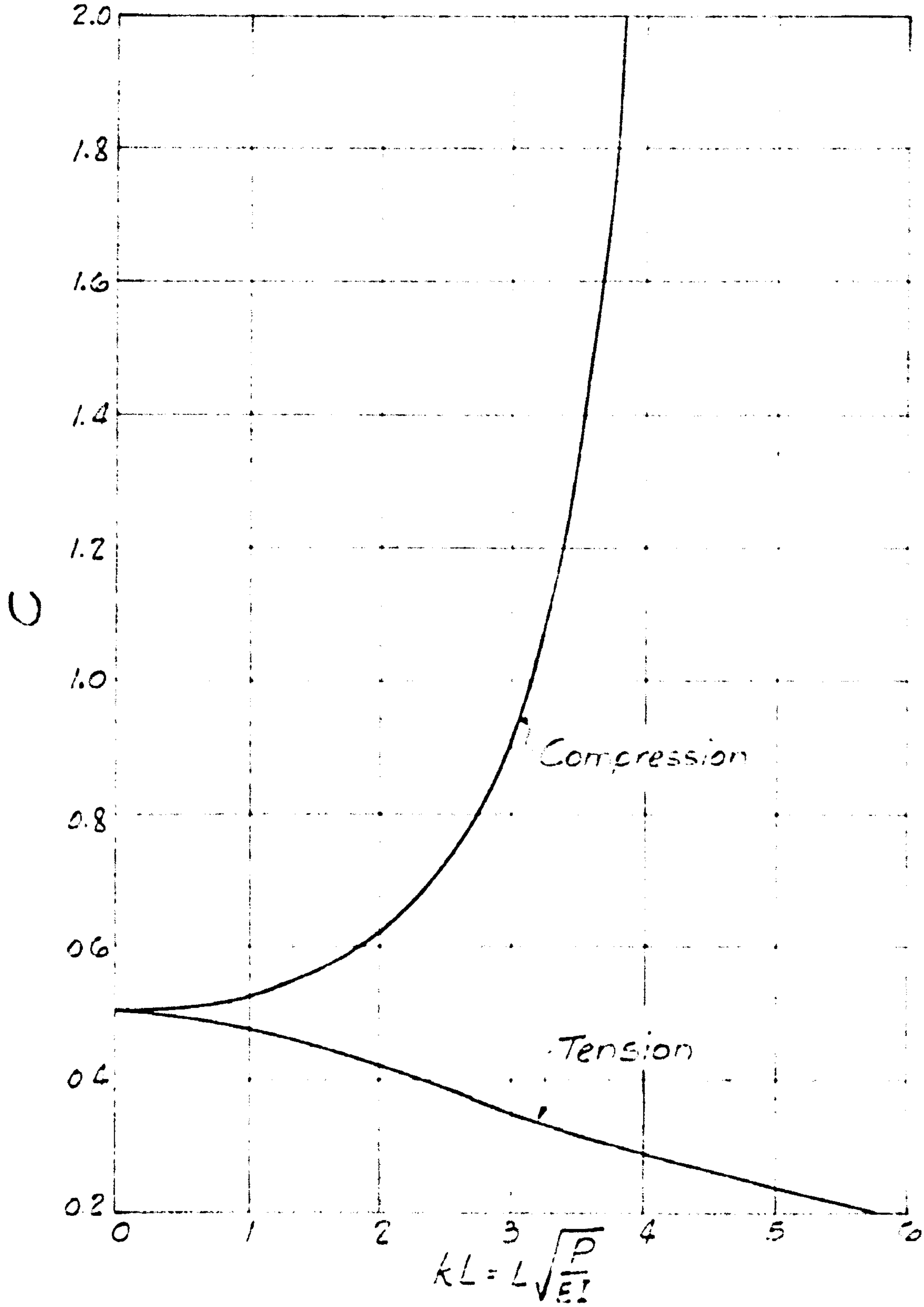

Figure 19. Carry-over factors for tension and compression members.

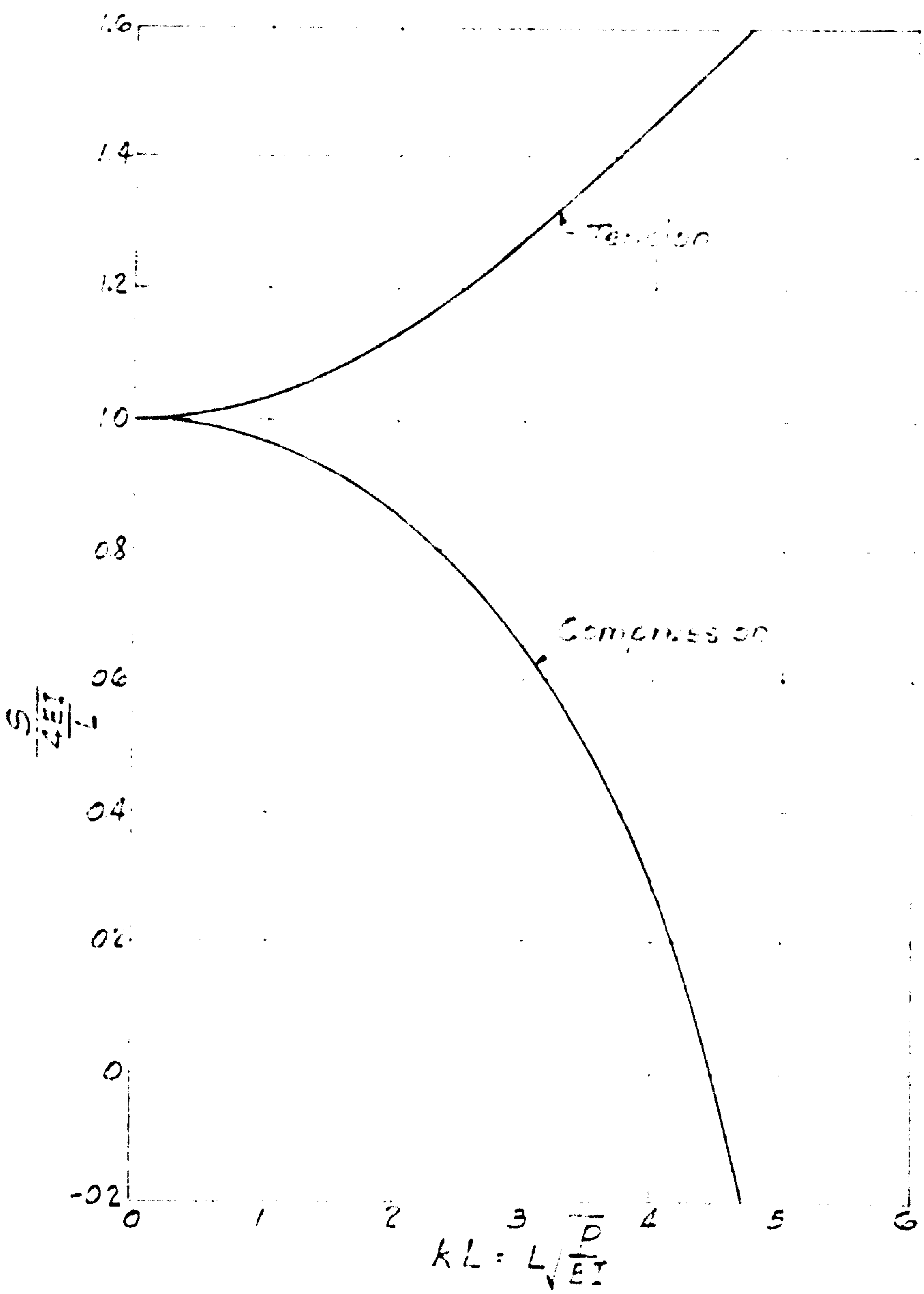

Figure 20. Stiffness factors for tension and compression members.

Appendix B

ILLUSTRATIVE EXAMPLE

The following example is introduced to demonstrate the use of the series and stiffness criteria, as presented in this thesis, to check the stability of a continuous beam-column shown in Figure 21(a). The joints A and E are hinged and fixed against deflection, joints B and D have elastic supports with a stiffness of 1.0 kip per inch against deflection, and joint C has an elastic support with a stiffness of 0.12 kips per inch against deflection. The individual members have a length of 100 inches, a moment of inertia of 10 in.^{4}, an axial stress of 100 kips compression, and a modulus of elasticity of 29,000 kips per sq. in.

The factors needed for the stability analysis using the series criterion are tabulated in Table 9. The quantities computed in intermediate calculations preceding the determination of the distribution and carry-over factors are also given in Table 9. Units of inches and kips were used in the calculations.

The first solution using the series criterion is shown in Table 10. An external moment of 100 inch kips was applied at joint A. All of the joints except A were balanced and then a summation was made of the carry-over moments at A. Only a part of the solution is shown since the joints had to be

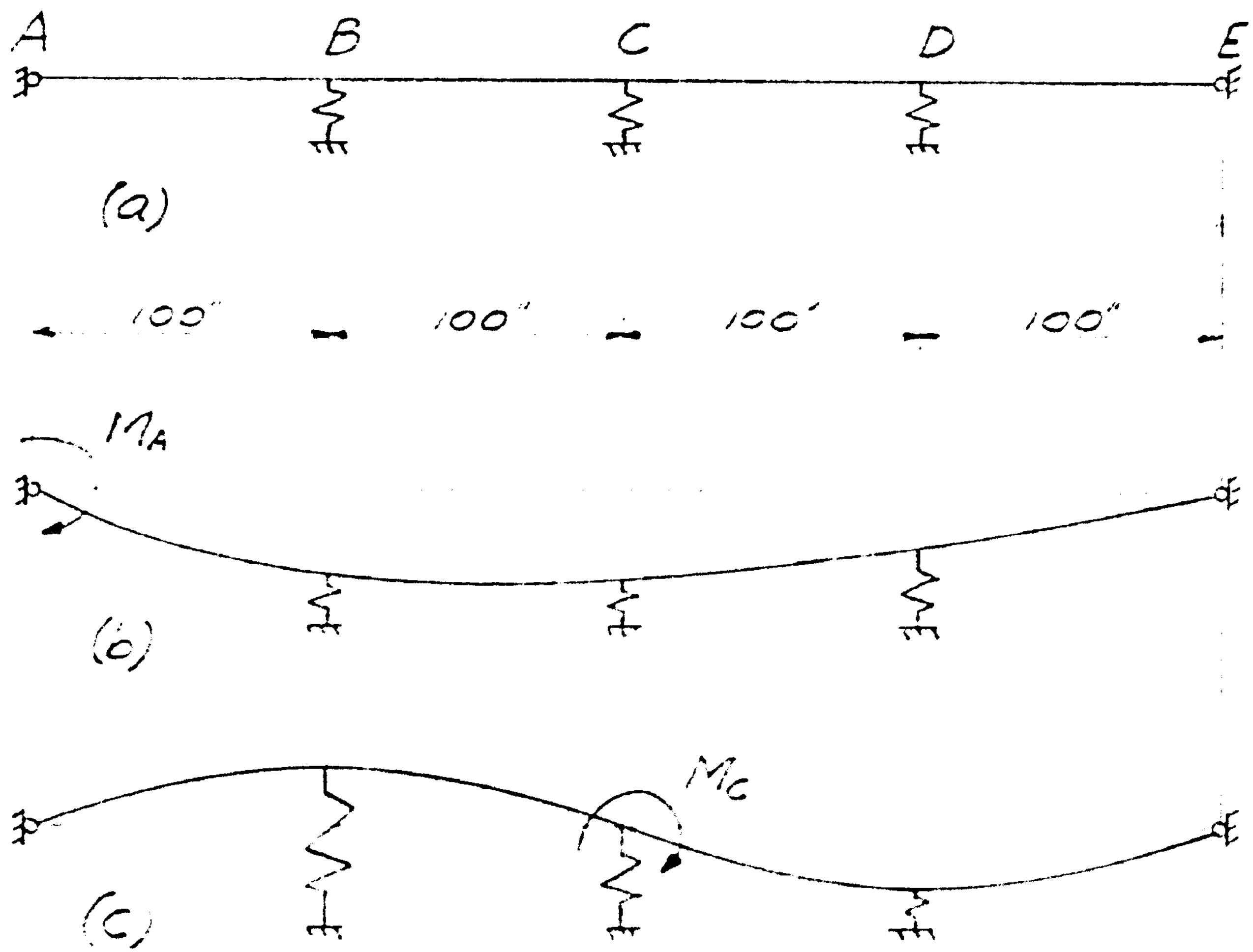

Figure 21. Elastically supported beam-column.

Table 9. Computed Quantities for Use in Applying the Series and Stiffness Criteria

t	A	B		C		D		E
		1.0		0.12		1.0		
C	0.605		0.605		0.605		0.605	
S	10,210		10,210		10,210		10,210	
S'''	16,390		16,390		16,390		16,390	
T	2.278		2.278		2.278		2.278	
t'		1.891	3.278	1.464	1.464	3.278	1.891	
T'	1.033	2.278	0.891	1.344	1.344	0.891	2.278	1.033
R	0.393	0.239	0.344	0.295	0.295	0.344	0.239	0.393
S	3,770	6,293	4,573	5,376	5,376	4,573	6,293	3,770
C_a	-0.0698	0.359	0.118	0.250	0.250	0.118	0.359	-0.0698
C_b	0.669		1.235	0.899	0.899	1.235		0.669
C_c			0.918	0.695	0.695	0.918		
D.F.	1.000	0.579	0.421	0.500	0.500	0.421	0.579	1.000

Table 10. Series Solution for a Moment Applied at A

	A		B		C		D	E
C_c	0		0.918	0.695	0.695	0.918		
C_b	0.669		1.235	0.899	0.899	1.235		0.669
C_a	-0.0698	0.359	0.118	0.250	0.250	0.118	0.359	-0.0698
D.F.	1.0	0.579	0.421	0.500	0.500	0.421	0.579	1.0
	+100	-6.98	+66.90	+66.90	+61.41	+61.41	+42.68	+42.68
	-12.45	-34.69	-25.23	-2.98	-31.16	-31.16	-21.66	-21.66
	-42.32	-42.32	-11.77	-47.08	-47.08	-11.77	-42.32	-42.32
	+1.02	+1.02	+1.47	+1.47	+0.14	+1.19	+1.63	+0.58
	+8.84	+8.84	+12.72	+12.72	+13.86	+13.86	-1.45	+20.72
	+6.24	+17.39	+12.65	+1.49	+15.62	+15.62	+10.86	+10.86
	-20.36	-20.36	-5.66	-22.65	-22.65	-5.66	-20.36	-20.36
	-4.65	-4.65	-6.69	-6.69	-0.64	-5.42	-7.45	-2.67
	+5.19	+5.19	+7.47	+7.47	+8.14	+8.14	-0.85	+12.17
	+5.13	+14.30	+10.40	+1.22	+12.84	+12.84	+8.92	+8.92
	-10.04	-10.04	-2.79	-11.17	-11.17	-2.79	-10.04	-10.04
	-5.87	-5.87	-8.44	-8.44	-0.81	-6.83	-9.39	-3.37
	+1.91	+1.91	+2.75	+2.75	+3.00	+3.00	-0.31	+4.49
	+4.67	13.02	+9.46	+1.12	+11.68	+11.68	+8.12	+8.12
	-4.18	-4.18	-1.16	-4.65	-4.65	-1.16	-4.18	-4.18
	-6.20	-6.20	-8.92	-8.92	-0.85	-7.22	-9.93	-3.56
	-0.16	-0.16	-0.23	-0.23	-0.25	-0.25	+0.03	-0.38
	etc.							
	$\sum = -100$							

balanced a large number of times to reach equilibrium. The factor r for this case was found to equal one, indicating a condition of neutral equilibrium; in other words, since no moment is required at A to rotate the member at joint A, a condition of instability exists. When r is found to be one, the moment distribution solution converges very slowly. When the value of r is greater or smaller than one, the moment distribution solution will converge more rapidly.

The deflection curve for the member with a moment applied at joint A would be similar to the curve shown in Figure 21(b).

A second solution using the series criterion is shown in Table 11. An external moment of 10,752 inch kips was applied at joint C. This magnitude of moment is necessary to rotate joint C one radian and is used so that the similarity with the stiffness criterion solution can be demonstrated. All of the joints except C were balanced and then a summation was made of the carry-over moments at C. This summation was found to be -8,750 inch kips indicating that a moment of 10,752 - 8,750 or 2,002 inch kips is required to rotate joint C one radian when the other joints have rotated and deflected to a position of equilibrium. The factor r for this case was found to equal 0.814 indicating a condition of stability. The member, however, is unstable as was shown in the first solution.

Table 11. Series Solution for a Moment Applied at C.

	A		B		C		D		E
C_c			0.918	0.695	0.695	0.918			
C_b	0.669		1.235	0.899	0.899	1.235		0.669	
C_a	-0.0698	0.359	0.118	0.250	0.250	0.118	0.359	-0.0698	
D.F.	1.0	0.579	0.421	0.5	0.5	0.421	0.579	1.0	
	+4833	+4833	+1344	+5376	+5376	+1344	+4833	+4833	
	-1284	-3576	-2601	-307	-3212	-3212	-2232	-2232	
	-3549	+248	-2374	-2374	-2179	-2179	-1514	-1514	
	+1070	+1070	+1539	+1539	+147	+1246	+1714	+615	
	-727	-727	-1046	-1046	-1139	-1139	+119	-1702	
	+268	+747	+543	+64	+671	+671	+466	+466	
	-611	+43	-409	-409	-375	-375	-261	-261	
	+187	+187	+269	+269	+26	+218	+301	+108	
	-133	-133	-192	-192	-209	-209	+22	-313	
	+49	+136	+99	+12	+122	+122	+85	+85	
	-103	+7	-69	-69	-63	-63	-44	-44	
	+32	+32	+46	+46	+4	+37	+50	+18	
	-25	-25	-36	-36	-39	-39	+4	-59	
	+9	+26	+19	+2	+23	+23	+16	+16	
	-16	+1	-11	-11	-10	-10	-7	-7	
	+4	+4	+6	+6	+1	+5	+8	+3	
	-5	-5	-7	-7	-8	-8	+1	-12	
	+3	+7	+5	+1	+6	+6	+4	+4	
	+2	0	+1	+1	+1	+1	0	0	
	-1	-1	-2	-2	0	-2	-2	-1	
	-1	-1	-2	-2	-2	-2	0	-3	

$\sum = -8750$

The deflection curve for the beam-column with a moment applied at joint C would be similar to the curve shown in Figure 21(c). Due to the symmetry of the beam-column, joint C would not deflect and is a node on the deflection curve.

The two solutions using the series criterion show the need for considering the shape of the deflection curve with its effect on the stability of the member.

The member shown in Figure 21(a) was also checked for stability using the stiffness criterion. The rotational and translational stiffnesses of joint C were determined; due to the symmetry of the member, the joint C is a node or a point of maximum deflection on the deflection curve respectively for the two stiffnesses. The rotational stiffness of joint E was determined, which agreed with the results of the first solution in applying the series criterion.

In using the equations derived for the stiffness criterion, and starting at the left end of the member, the results are as follows:

Member AB,

$$S_{Bzz} = 6,470$$

$$S_{Byz} = 64.7$$

$$S_{Byy} = 0.647$$

Member BC,

$$S_{Czz} = 1,000$$

$$S_{Cyz} = 32.2$$

$$S_{Cyy} = 0.060$$

Since the entire member is symmetrical about joint C, the rotational stiffness of joint C is 1,000 + 1,000 = 2,000 inch kips. This moment agrees in magnitude with the value obtained in the second solution using the series criterion. The value of the rotational stiffness indicates that the member is stable if it is deflected with a node at joint C.

The translational stiffness of joint C is 0.060 + 0.060 - 0.120 = 0 kips per inch. The stiffness of the elastic support at C is included in each of the 0.060 values and therefore the 0.120 value must be subtracted to obtain the true translational stiffness of the joint C considering the entire structure on both sides of joint C. The value of the translational stiffness indicates that the member is unstable since no force is required to deflect joint C. The member, therefore, would buckle as shown in Figure 21(b).

Continuing the solution and working to the right from joint C, the results are as follows:

Member CD,

$$S_{Dzz} = -3,740$$
$$S_{Dyz} = 32.8$$
$$S_{Dyy} = 0.725$$

Member DE,

$$S_{Ezz} = 0$$

The rotational stiffness of zero at joint E agrees with the first solution made using the series criterion and also indicates that the member is unstable.

Appendix C

TORSIONAL STIFFNESS OF THE END POST MEMBER

The torsional stiffness of the end post member was required to determine the elastic properties, using equations (132) and (137), of the horizontal projection of the member.

The torsional stiffness, S_t, is defined as the torque per unit of twist required to twist a member. The procedure used to evaluate the torsional stiffness has been presented by Hrennikoff (17).

A cross section of the end post member is shown in Figure 22. Portions of the channel flanges and of the cover plate outside of the gage lines of the rivets do not contribute to the torsional stiffness and are not shown on the figure.

When a torque is applied to the end post member, shearing stresses are developed as shown in Figure 22. The shear flow, q, is a constant for a thin-walled section. The shear flow is defined as the force per unit of length of periphery of the cross section. Let τ represent the shearing unit stress, then,

$$q = 0.24\ \tau_1 = 0.375\ \tau_2 = 0.436\ \tau_3 \qquad (140)$$

The force, P, in the lacing bar can be expressed as

$$P \cos 30 = (0.375)\ \tau_2 (14.25)$$

$$P = 6.17\ \tau_2 \qquad (141)$$

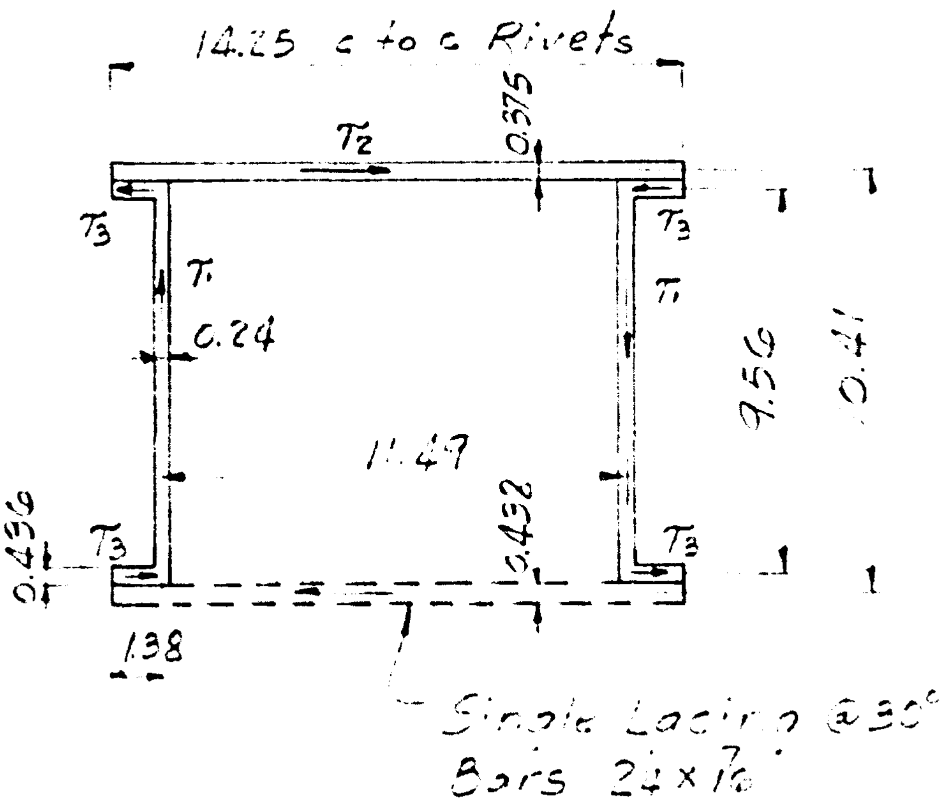

Figure 22. Cross section of end post member.

The total torque developed by the cross section can be expressed as

$$M_t = (0.24)\tau_1(9.56)(11.49) + (0.375)\tau_2(14.25)(10.41) - (2)(0.436)\tau_3(1.38)(9.56) \quad (142)$$

Equating the external and internal work per unit of length of the member results in

$$\frac{M_t^2}{2GJ_e} = \frac{\tau_1^2(0.24)(9.56)(2)}{2G} + \frac{\tau_2^2(0.375)(14.25)}{2G} + \frac{\tau_3^2(0.436)(1.38)(4)}{2G} + \frac{P^2(2.0)}{2E(2.25)(0.438)} \quad (143)$$

By substituting the relationships from equations (140), (141) and (142) in equation (143), the equivalent polar moment of inertia, J_e, can be found. Using the relationship $G = 0.4E$,

$$J_e = 153.4 \text{ in.}^4$$

The torsional stiffness, S_t, for the end post member is, therefore,

$$S_t = \frac{GJ_e}{L}$$

Assuming $G = 11{,}600$ kips per sq. in.,

$$S_t = 7{,}595 \text{ inch kips per radian of twist.}$$